LAST NAME: Key FIRST NAME: Quiz 2

MATH 602, Differential Equations Prof: Dr. Adam Larios

Notes, books, and calculators are not authorized. Show all your work in the blank space you are given. Always justify your answer. Answers without adequate justification will not receive credit.

1. **(2 points)** Suppose L is a linear operator, and that $L(u) = f$, $L(v) = g$. Compute the following expressions in terms of f and g, or state that it is not possible.

$L(3u - 7v) = 3L(u) - 7L(v) = 3f - 7g$

$L(u^2) =$ not possible; for linear operators, $L(u^2) \neq L(u) \cdot L(u)$ in general.

2. **(6 points)** Find all possible functions ϕ and real numbers λ that satisfy the following eigenvalue problem.

$$\begin{cases} \phi'' = -\lambda\phi, \\ \phi(0) = 0, & \text{BC 1} \\ \phi'(L) = 0. & \text{BC 2} \end{cases}$$

Case I, $\lambda = 0$

$\phi'' = 0 \Rightarrow \phi = ax + b$

BC 1: $0 = \phi(0) = b \Rightarrow b = 0$

BC 2: $0 = \phi'(L) = a \Rightarrow a = 0$

Thus, $\boxed{\phi = 0}$

Case II, $\lambda < 0$

Let $s = -\lambda > 0$.

Then $\phi'' = s\phi$, so general solution is

$\phi(x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$.

BC 1: $0 = \phi(0) = c_1 + c_2 \Rightarrow c_1 = -c_2$

And $\phi'(x) = \sqrt{s}c_1 e^{\sqrt{s}x} - c_2\sqrt{s}e^{-\sqrt{s}x}$.

Now, since $c_1 = -c_2$,

$\phi'(x) = \sqrt{s}c_1 \left(e^{\sqrt{s}x} + e^{-\sqrt{s}x}\right)$

BC 2: $0 = \phi'(L) = \sqrt{s}c_1\left(e^{\sqrt{s}L} + e^{-\sqrt{s}L}\right)$

$\Rightarrow c_1 = 0$ since $s \neq 0$, and $e^{\sqrt{s}L} > 0$ and $e^{-\sqrt{s}L} > 0$

and thus,

$c_2 = -c_1 = -0 = 0$,

So $\boxed{\phi = 0}$

Case III, $\lambda > 0$

$\phi'' = -\lambda\phi \Rightarrow$

$\phi(x) = c_3 \sin(\sqrt{\lambda}x) + c_4 \cos(\sqrt{\lambda}x)$

BC 1:

$0 = \phi(0) = c_3 \sin(0) + c_4 \cos(0) = c_4$

So $\phi(x) = c_3 \sin(\sqrt{\lambda}x)$

and $\phi'(x) = c_3\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

BC 2:

$0 = \phi'(L) = c_3\sqrt{\lambda}\cos(\sqrt{\lambda}L)$

since $\lambda \neq 0$ and to have a non-trivial solution we want $c_3 \neq 0$, we must have $\sqrt{\lambda}L = (n - \frac{1}{2})\pi$, $n = 1, 2, 3, \dots$, that is

$$\boxed{\lambda = \frac{(n-\frac{1}{2})\pi}{L}, n = 1, 2, 3, \dots}$$

and

$$\boxed{\phi(x) = C \cdot \sin\left(\frac{(n-\frac{1}{2})\pi}{L}\right)}$$

TURN OVER

Q2

3. **(4 points)** Consider the following heat equation in variables $x \in [0, L]$ and $t > 0$.

$$\begin{cases} \partial_t T - k\partial_{xx} T = 0, \\ T(0,t) = 0, \\ -k\partial_n T(L,t) = 0, \end{cases}$$

where $k > 0$ is a constant. Suppose $T(x,t) = \phi(x)G(t)$ Use the method of separation of variables to write down ODEs (ordinary differential equations) for ϕ and G.

Plugging $T = \phi G$ into the equation,

$$\partial_t(\phi G) - k\,\partial_{xx}(\phi G) = 0$$

$$\Rightarrow \quad \phi(x)\frac{\partial G}{\partial t}(t) = k\,G(t)\frac{\partial^2}{\partial x^2}\phi(x)$$

Divide by $k\,\phi(x)G(t)$:

$$\underbrace{\frac{1}{k\,G(t)}\frac{\partial G(t)}{\partial t}}_{\text{function only of } t} = \underbrace{\frac{1}{\phi(x)}\frac{\partial^2\phi(x)}{\partial x^2}}_{\text{function only of } x} = -\lambda$$

← Therefore both sides must equal a constant, say $-\lambda$ (λ is also ok)

Thus

$$\partial_t G = -k\lambda G \quad \text{and} \quad \phi_{xx} = -\lambda\phi$$

<u>Note:</u> It is not necessary for this problem, but we can put boundary conditions on ϕ, namely $\begin{cases} \phi(0) = 0, \\ \phi'(L) = 0. \end{cases}$

4. **(3 points)** Consider Problem 3, above, with $L = \pi$ and initial data

$$T(x,0) = \sin(2.5x).$$

Find the solution $T(x,t)$. [Hint: Use results you calculated on this quiz already.]

Solving $\partial_t G = -k\lambda G$ gives $G(t) = c_4 e^{-k\lambda t}$ for some constant c_4.

From problem 2, $\phi = c_3 \sin\left(\left(n-\frac{1}{2}\right)\frac{\pi x}{L}\right) \overset{L=\pi}{=} c_3 \sin\left(\left(n-\frac{1}{2}\right)x\right)$

$\lambda = \left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2 = \ldots$

Thus, we have solutions of the form $T(x,t) = c\,e^{-k\lambda t}\sin\left(\left(n-\frac{1}{2}\right)x\right)$ (with $c = c_3 c_4$, say). Choosing $c=1$, $n=3$, we can match the initial condition at $t=0$. Thus,

$$\boxed{T(x,t) = e^{-k(2.5)^2 t}\sin(2.5x)}$$