FULL NAME: Key (please print) Exam 1
MATH 221H, Differential Equations Dr. Adam Larios No calculators

Answers without full, proper justification will not receive full credit.

1. **(12 points)** Solve the following initial value problem.

$$\begin{cases} \dfrac{dy}{dt} = y^{-2}\cos(t), \\ y(0) = 2. \end{cases}$$ ← Nonlinear, but separable

Separate:

$$y^2\,dy = \cos(t)\,dt$$

Integrate:

$$\int y^2\,dy = \int \cos(t)\,dt$$

$$\Rightarrow \frac{1}{3}y^3 = \sin(t) + C$$

$$\Rightarrow y(t) = (3\sin(t) + 3C)^{1/3}$$

Solve for C:

$$2 = y(0) = (3\sin(0) + 3C)^{1/3}$$

$$\Rightarrow 2 = (3C)^{1/3}$$

$$\Rightarrow 8 = 3C$$

$$\Rightarrow \frac{8}{3} = C$$

$$\Rightarrow \boxed{y(t) = (3\sin(t) + 8)^{1/3}}$$

2. **(12 points)** Solve the following equation up to an arbitrary constant c.

$$\frac{dy}{dt} + \frac{4}{t}y = \frac{1}{t^3} - 1$$ Linear, but not separable. Use integrating factor.

$$\mu(t) = e^{\int \frac{4}{t}dt} = e^{4\ln|t|} = e^{\ln(|t|^4)} = |t|^4 = t^4$$

Cannot cancel exponent and log yet.

Multiply by $\mu(t) = t^4$.

$$t^4\frac{dy}{dt} + 4t^3 y = t - t^4$$

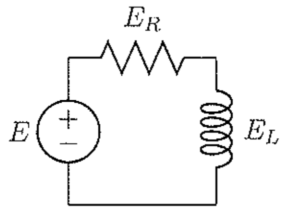
$$\Rightarrow \frac{d}{dt}(t^4 y) = t - t^4$$

$$\Rightarrow t^4 y = \frac{1}{2}t^2 - \frac{1}{5}t^5 + C$$

Cannot absorb t^{-4} into constant.

$$\Rightarrow \boxed{y(t) = \frac{1}{2}t^{-2} + \frac{1}{5}t^3 + ct^{-4}}$$

3. **(6 points)** Consider the following LR-circuit.



Inductance: $L = 0.3$ [Henrys],
Resistance: $R = 5$ [Ohms],
Supplied Voltage: $E = E(t) = 3\sin(60t)$

The voltage drops across these components are given in terms of the current $I = I(t)$, by $E_R = RI$ (for the resistor) and $E_L = L\frac{dI}{dt}$ (for the inductor). Kirchoff's loop rule says that the supplied voltage E must be balanced by the voltage drops across each component. Write down a differential equation for this circuit.

Balance: $E_L + E_R = E$ $\Rightarrow$ $L\frac{dI}{dt} + RI = 3\sin(60t)$

$\Rightarrow$ $\boxed{0.3\frac{dI}{dt} + 5I = 3\sin(60t)}$

4. **(12 points)** Consider the equation: $x^2\frac{dy}{dx} = xy + y^2$.

Solve the equation up to a constant c. (**HINT:** Use the substitution $v = \frac{y}{x}$, or $y = xv$.)

Divide by x^2:

$$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

$$\Rightarrow \frac{dy}{dx} = v + v^2$$

Have $\frac{dy}{dx}$, need $\frac{dv}{dx}$.

Get $\frac{dv}{dx}$ from $v = \frac{y}{x}$ or $y = xv$.

Let's use $y = xv$. Take $\frac{d}{dx}$:

product rule

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

Set equal:

$$x\frac{dv}{dx} + v = \frac{dy}{dx} = v + v^2$$

Thus,

$$x\frac{dv}{dx} = v^2$$

Separate:

$$\int v^{-2}\,dv = \int x^{-1}\,dx$$

$$\Rightarrow \frac{v^{-1}}{-1} = \ln|x| + c$$

Solve for v:

$$v = -(\ln|x| + c)^{-1}$$

Substitute $v = \frac{y}{x}$ and

Solve for y:

$$y = -x(\ln|x| + c)^{-1}$$

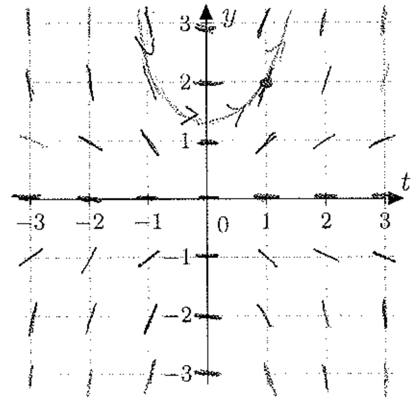
or

$$\boxed{y = \frac{-x}{\ln|x| + c}}$$

Note: there are other solutions, such as $y = 0$.

5. **(8 points)** Plot a **direction field** as completely as possible for the given equation.
Sketch the **integral curve** that passes through the point $(1, 2)$.
There is no need to solve the equation.

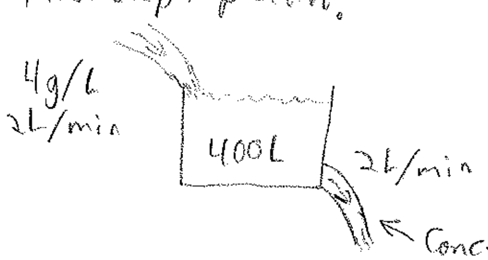
$\boxed{y' = ty}$



← Drawing this shape is cool.

6. (a) **(12 points)** A tank initially contains **400L of pure water**. Water with a concentration of **4g/L** of salt is then pumped into the tank at the rate of **2L/min**, and the well-stirred mixture leaves at the same rate. **How long** does it take for the concentration of salt in the tank to become **1g/L**? **(You do not need to find the decimal value.)**

First step: picture!



← Concentration out: $\frac{A}{400L}$

$A(t)$ = The amount of salt in the tank (in grams) at time t (in minutes)

$$\frac{dA}{dt} = \left(\frac{4g}{L}\right)\left(\frac{2L}{min}\right) - \left(\frac{A}{400L}\right)\left(\frac{2L}{min}\right) = 8 - \frac{1}{200}A$$

Separable & linear, so solve by either method.
Let's use separable (this is the long way):

$$\int \frac{1}{8 - \frac{1}{200}A}\, dA = \int dt$$

$$\Rightarrow \frac{1}{-1/200} \ln\left|8 - \frac{1}{200}A\right| = t + C_1$$

$$\Rightarrow \ln\left|8 - \frac{1}{200}A\right| = -\frac{1}{200}t + C_2$$

$$\Rightarrow \left|8 - \frac{1}{200}A\right| = C_3 e^{-t/200}$$

$$\Rightarrow 8 - \frac{1}{200}A = C_4 e^{-t/200}$$

$$\Rightarrow 1600 - A = C_5 e^{-t/200}$$

$$\Rightarrow A(t) = 1600 - C_5 e^{-t/200}$$

Initially pure $\Rightarrow A(0) = 0$, so

$$0 = A(0) = 1600 - C_5 e^0 \Rightarrow C_5 = 1600$$

$$\Rightarrow A(t) = 1600 - 1600 e^{-t/200}$$

Concentration @ 1g/L $\Rightarrow (400L)\frac{1g}{L} = 400g$

$$400 \overset{set}{=} A(t) = 1600 - 1600 e^{-t/200}$$

$$\Rightarrow \frac{3}{4} = e^{-t/200} \Rightarrow \frac{-t}{200} = \ln\left(\frac{3}{4}\right)$$

$$\Rightarrow \boxed{t = -200 \ln\left(\frac{3}{4}\right)} \approx 57.36 \text{ min}$$

(b) **(2 points)** What is the **time-constant** of this model (i.e., the "characteristic time" for which the solution "changes significantly")?

$$k = \frac{1}{1/200} = 200 \text{ min}$$

(see definition in book & your notes)

7. **(12 points)** Consider the problem $\begin{cases} y' = ty - 2, \\ y(3) = 2. \end{cases}$

Use the **forward Euler** method with step size $\Delta t = 0.5$ to approximate $y(4)$.

Hint: Think about what t_0 and y_0 are before you begin.

forward Euler ← sounds like "oiler" !!

$$y_{n+1} = y_n + \Delta t\, f(t_n, y_n)$$

$t_0 = 3$, $y_0 = 2$

$\Delta t = 0.5$, $t_0 = 3$, so

$y_0 = y(3)$

$y_1 \approx y(3.5)$

$y_2 \approx y(4.0)$

$y_3 \approx y(4.5)$ ← too far!

$$y_1 = y_0 + \Delta t\, f(t_0, y_0)$$

$$\Rightarrow y_1 = 2 + (0.5)\big((3)(2) - 2\big) = 4$$

and $t_1 = t_0 + \Delta t = 3 + 0.5 = 3.5$

Then

$$y_2 = y_1 + \Delta t\, f(t_1, y_1)$$

$$\Rightarrow y_2 = 4 + (0.5)\big((3.5)(4) - 2\big) = 10$$

8. The trapezoidal rule gives a numerical method for solving $y' = f(t, y)$. It is given by

$$y_{n+1} = y_n + h\left(\frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}\right)$$

where $h = \Delta t = t_{n+1} - t_n$ is the step size.

(a) **(2 points)** Is this method implicit or explicit?

implicit (must solve for y_{n+1})

(b) **(6 points)** Consider solving the equation $y' = 2ty$ via the trapezoidal rule given above. Find a formula for y_{n+1} involving **only** h, y_n, t_n, and t_{n+1}.

$$y_{n+1} = y_n + h\left(\frac{\cancel{2} t_n y_n + \cancel{2} t_{n+1} y_{n+1}}{\cancel{2}}\right) = y_n + h t_n y_n + h t_{n+1} y_{n+1}$$

$$\Rightarrow y_{n+1} - h t_{n+1} y_{n+1} = y_n + h t_n y_n$$

$$\Rightarrow (1 - h t_{n+1}) y_{n+1} = (1 + h t_n) y_n$$

$$\Rightarrow \boxed{y_{n+1} = \left(\frac{1 + h t_n}{1 - h t_{n+1}}\right) y_n}$$

9. In class, we modeled a population $P = P(t)$ with carrying capacity $K > 0$ by the logistic equation, that is,

$\frac{dP}{dt} = rP(1 - \frac{P}{K})$

where $r > 0$ is the intrinsic growth rate. Suppose a population $P(t)$ has a carrying capacity of $K = 200$, and an intrinsic growth rate of $r = 0.03$, **but it also has a "threshold,"** where the population decreases if it is below 20.

(a) **(8 points)** Write down a model describing this population. (**Hint:** Think about signs.)

$$\frac{dP}{dt} = 0.03P\left(1 - \frac{P}{200}\right)\left(\frac{P}{20} - 1\right) \quad \text{negative when } P < 20$$

(b) **(4 points)** Is your equation linear or nonlinear? What mathematical method could you use to solve it?

nonlinear, but it is separable (in fact, autonomous)

So use method of separable equations (i.e. divide by entire right-hand side).

(c) **(4 points)** For which values of P will the population not change in time? (These are called "equilibrium values.")

Equilibria:

$$0 \overset{\text{set}}{=} \frac{dP}{dt}$$

$$0 = 0.03P\left(1 - \frac{P}{200}\right)\left(\frac{P}{20} - 1\right)$$

So either $P = 0$ or $1 - \frac{P}{200} = 0$ or $\frac{P}{20} - 1 = 0$,

So $P = 0$ or 200 or 20.