A New Proof of Mercer's Extension Theorem

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Mercer's Extension Theorem

Theorem (Mercer '91)

Let $\mathcal{A}_i \subseteq (\mathcal{M}_i, \mathcal{D}_i)$ *be Cartan bimodule algebras and* $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ *be a Cartan bimodule isomorphism. Then there exists a* $\star$*-isomorphism* $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ *such that* $\pi|_{\mathcal{A}_1} = \theta$.

Normalizers

Let $\mathcal{C}$ be a unital $C^\star$-algebra and $\mathcal{D} \subseteq \mathcal{C}$ be a unital abelian $C^\star$-subalgebra. Then

- $UN(\mathcal{C}, \mathcal{D}) = \{u \in U(\mathcal{C}) : u\,\mathcal{D}\,u^\star = \mathcal{D}\}$
- $GN(\mathcal{C}, \mathcal{D}) = \{v \in \mathcal{C} : v$ is a partial isometry and $v\,\mathcal{D}\,v^\star, v^\star\,\mathcal{D}\,v \subseteq \mathcal{D}\}$
- $N(\mathcal{C}, \mathcal{D}) = \{x \in \mathcal{C} : x\,\mathcal{D}\,x^\star, x^\star\,\mathcal{D}\,x \subseteq \mathcal{D}\}$

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- $N(\mathcal{C}, \mathcal{D}) = \{x \in \mathcal{C} : x\,\mathcal{D}\,x^\star, x^\star\,\mathcal{D}\,x \subseteq \mathcal{D}\}$

Note:

- $UN(\mathcal{C}, \mathcal{D})$ is a group
- $GN(\mathcal{C}, \mathcal{D})$ and $N(\mathcal{C}, \mathcal{D})$ are $\star$-semigroups
- $U(\mathcal{D}) \subseteq UN(\mathcal{C}, \mathcal{D}) \subseteq GN(\mathcal{C}, \mathcal{D}) \subseteq N(\mathcal{C}, \mathcal{D})$

Cartan Subalgebras

Definition (Cartan subalgebra)

Let $\mathcal{M}$ be a von Neumann algebra. We say that $\mathcal{D} \subseteq \mathcal{M}$ is a **Cartan subalgebra** if the following conditions hold:

1. $\mathcal{D}$ is a MASA in $\mathcal{M}$.
2. $\overline{span}^{\sigma}(UN(\mathcal{M}, \mathcal{D})) = \mathcal{M}$.
3. There exists a normal faithful conditional expectation $\mathbb{E} : \mathcal{M} \to \mathcal{D}$.

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Example

$D_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ is a Cartan subalgebra. Indeed,

$$UN(M_n(\mathbb{C}), D_n(\mathbb{C})) = \{PV : P \in M_n(\mathbb{C}) \text{ permutation matrix, } V \in D_n(\mathbb{T})\}.$$

The Feldman-Moore Construction ('77)

Inputs:

- X, a standard Borel space
- R, a countable Borel equivalence relation on X
- μ, a probability measure on X which is quasi-invariant for R
- s, a normalized 2-cocycle on R

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Output:

- ν, right counting measure on R relative to μ
- $L^2(R,\nu)$, a separable Hilbert space
- $\mathbf{M}(R,s) \subseteq B(L^2(R,\nu))$, a von Neumann algebra consisting of certain bounded Borel functions $T : R \to \mathbb{C}$ acting on $L^2(R,\nu)$ by twisted matrix multiplication:

$$T\xi(x,y) = \sum_{zRx} T(x,z)\xi(z,y)s(x,z,y), \ \xi \in L^2(R,\nu), \ (x,y) \in R$$

- $\mathbf{D}(R,s) = \{T \in \mathbf{M}(R,s) : T(x,y) = 0 \text{ if } x \neq y\}$, a Cartan subalgebra of $\mathbf{M}(R,s)$

The Feldman-Moore Representation Theorem

Theorem (Feldman, Moore '77)

Let $\mathcal{M}$ be a von Neumann algebra with separable predual and $\mathcal{D} \subseteq \mathcal{M}$ be a Cartan subalgebra. Then there exists X, R, μ, and s such that $\mathcal{M} \cong \mathbf{M}(R, s)$, with $\mathcal{D} \cong \mathbf{D}(R, s)$.

Cartan Bimodule Algebras

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Let $\mathcal{M}$ be a von Neumann algebra with separable predual and $\mathcal{D} \subseteq \mathcal{M}$ be a Cartan subalgebra. We say that $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{M}$ is a **Cartan bimodule algebra** if the following conditions hold:

1. $\mathcal{A}$ is a σ-weakly closed (non-self-adjoint) subalgebra.
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Example

$$D_4(\mathbb{C}) \subseteq \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} : a_{ij} \in \mathbb{C} \right\} \subseteq M_4(\mathbb{C})$$

is a Cartan bimodule algebra.

The Spectral Theorem for Bimodules

Theorem (Muhly, Saito, Solel '88)

Let $\mathcal{A} \subseteq (\mathcal{M}, \mathcal{D})$ *be a Cartan bimodule algebra. Then there exists a unique Borel set* $\Gamma(\mathcal{A}) \subseteq R$ *such that*

$$\mathcal{A} \cong \{T \in \mathbf{M}(R, s) : T(x, y) = 0 \textit{ for all } (x, y) \notin \Gamma(A)\}.$$

In fact, $\Gamma(\mathcal{A})$ *is a reflexive and transitive relation which generates* R*.*

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In fact, $\Gamma(\mathcal{A})$ is a reflexive and transitive relation which generates R.

Corollary (abundance of normalizers)

Let $\mathcal{A} \subseteq (\mathcal{M}, \mathcal{D})$ be a Cartan bimodule algebra. Then

$$\overline{span}^{\sigma}(GN(\mathcal{A}, \mathcal{D})) = \mathcal{A}.$$

Cartan Bimodule Isomorphisms

Definition (Cartan bimodule isomorphism)

Let $\mathcal{A}_i \subseteq (\mathcal{M}_i, \mathcal{D}_i)$, $i = 1, 2$, be Cartan bimodule algebras. We say that $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ is a **Cartan bimodule isomorphism** if the following conditions hold:

1. θ is an isometric isomorphism.
2. $\theta(\mathcal{D}_1) = \mathcal{D}_2$.

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Example

Let $\alpha, \beta, \gamma \in \mathbb{R}$. Then

$$\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & e^{i\alpha} a_{12} & 0 & e^{i\beta} a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & e^{i\gamma} a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

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Mercer's Representation Theorem

Theorem (Mercer '91)

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1. $(\tau \times \tau)(R_1) = R_2$.
2. $(\tau \times \tau)(\Gamma(\mathcal{A}_1)) = \Gamma(\mathcal{A}_2)$.
3. $\theta(T)(x, y) = m(x, y) T(\tau^{-1}(x), \tau^{-1}(y)),\ \ T \in \mathcal{A}_1,\ \ (x, y) \in \Gamma(\mathcal{A}_2)$.

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$$\begin{bmatrix} a_{11} & e^{i\alpha}a_{12} & 0 & e^{i\beta}a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & e^{i\gamma}a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & e^{i\alpha} & 0 & e^{i\beta} \\ 0 & 1 & 0 & 0 \\ 0 & e^{i\gamma} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

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Proof.

Extend $m : \Gamma(\mathcal{A}_2) \to \mathbb{T}$ in Mercer's Representation Theorem to $\overline{m} : R_2 \to \mathbb{T}$ in an appropriate way and define

$$\pi(T)(x, y) = \overline{m}(x, y) T(\tau^{-1}(x), \tau^{-1}(y)), \quad T \in \mathcal{M}_1, \ (x, y) \in R_2.$$

□

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Extend $m : \Gamma(\mathcal{A}_2) \to \mathbb{T}$ in Mercer's Representation Theorem to $\overline{m} : R_2 \to \mathbb{T}$ in an appropriate way and define

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Example

$$m = \begin{bmatrix} 1 & e^{i\alpha} & 0 & e^{i\beta} \\ 0 & 1 & 0 & 0 \\ 0 & e^{i\gamma} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \overline{m} = \begin{bmatrix} 1 & e^{i\alpha} & e^{i(\alpha-\gamma)} & e^{i\beta} \\ e^{-i\alpha} & 1 & e^{-i\gamma} & e^{-i(\alpha-\beta)} \\ e^{-i(\alpha-\gamma)} & e^{i\gamma} & 1 & e^{-i(\alpha-\beta-\gamma)} \\ e^{-i\beta} & e^{i(\alpha-\beta)} & e^{i(\alpha-\beta-\gamma)} & 1 \end{bmatrix}$$

On the other hand...

$$\begin{bmatrix} a_{11} & e^{i\alpha}a_{12} & 0 & e^{i\beta}a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & e^{i\gamma}a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} = U \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} U^\star,$$

where

$$U = \begin{bmatrix} e^{i\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\gamma} & 0 \\ 0 & 0 & 0 & e^{i(\alpha-\beta)} \end{bmatrix}.$$

Norming Subalgebras

Definition (Pop, Sinclair, Smith '00)

Let $\mathcal{A}$ be a unital operator algebra and $\mathcal{D} \subseteq \mathcal{A}$ be a unital $C^\star$-subalgebra. We say that $\mathcal{D}$ norms $\mathcal{A}$ if for all $X \in M_n(\mathcal{A})$, we have that

$$\|X\| = \sup\{\|RXC\| : R \in \mathrm{Ball}(M_{1,n}(\mathcal{D})),\ C \in \mathrm{Ball}(M_{n,1}(\mathcal{D}))\}.$$

Norming Subalgebras

Theorem (Pop, Sinclair, Smith '00)

1. *Any unital $C^\star$-algebra norms itself.*
2. *Any MASA norms $B(\mathcal{H})$.*
3. *A unital $C^\star$-algebra is normed by the scalars if and only if it is abelian.*
4. *If $\mathcal{N} \subseteq \mathcal{M}$ is a finite-index inclusion of II_1 factors, then $\mathcal{N}$ norms $\mathcal{M}$.*

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Theorem (Sinclair, Smith '98)

Let $\mathcal{A} \subseteq \mathcal{M} \subseteq B(\mathcal{H})$ be von Neumann algebras. Suppose there exists an abelian von Neumann algebra $\mathcal{B} \subseteq \mathcal{M}'$ such that $C^\star(\mathcal{A}, \mathcal{B})$ is cyclic. Then $\mathcal{A}$ norms $\mathcal{M}$.

Cartan Subalgebras are Norming

Corollary (Cameron, Pitts, Z.)

Let $\mathcal{M}$ be a von Neumann algebra with separable predual and $\mathcal{D} \subseteq \mathcal{M}$ be a Cartan subalgebra. Then $\mathcal{D}$ norms $\mathcal{M}$.

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Proof.

From Feldman-Moore, $\mathbf{M}(R, s)$ has a cyclic and separating vector, and so there exists an anti-unitary $J : L^2(R, \nu) \to L^2(R, \nu)$ such that $J\mathbf{M}(R, s)J = \mathbf{M}(R, s)'$. Also $W^\star(\mathbf{D}(R, s), J\mathbf{D}(R, s)J)$ is a MASA in $B(L^2(R, \nu))$, and therefore is cyclic, so that $C^\star(\mathbf{D}(R, s), J\mathbf{D}(R, s)J)$ is cyclic as well. By Sinclair and Smith, $\mathbf{D}(R, s)$ norms $\mathbf{M}(R, s)$. □

Pitts' Automatic Complete-Boundedness Theorem

Theorem (Pitts '08)

Let $\mathcal{A}$ and $\mathcal{B}$ be operator algebras and $\theta : \mathcal{A} \to \mathcal{B}$ be a bounded isomorphism. If $\mathcal{B}$ contains a norming $C^\star$-subalgebra, then θ is completely bounded and

$$\|\theta\|_{cb} \leq \|\theta\|\|\theta^{-1}\|^4.$$

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Remark

Pitts' theorem relies crucially on the following remarkable theorem of Pisier/Haagerup ('78/'83): Let $\mathcal{C}$ be a $C^\star$-algebra and $\rho : \mathcal{C} \to B(\mathcal{H})$ be a bounded homomorphism. Then

$$\|\rho_{n,1}(C)\| \leq \|\rho\|^2 \|C\|, \;\; C \in M_{n,1}(\mathcal{C})$$

and

$$\|\rho_{1,n}(R)\| \leq \|\rho\|^2 \|R\|, \;\; R \in M_{1,n}(\mathcal{C}).$$

The Unique Pseudo-Expectation Property (Pitts, Z.)

Definition

Let $\mathcal{C}$ be a unital $C^\star$-algebra and $\mathcal{D} \subseteq \mathcal{C}$ be a unital $C^\star$-subalgebra. We say that $(\mathcal{C}, \mathcal{D})$ has the **unique pseudo-expectation property** if there exists a unique ucp map $\mathbb{E} : \mathcal{C} \to I(\mathcal{D})$ such that $\mathbb{E}|_{\mathcal{D}} = \mathrm{id}$.

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Example

$(B(\ell^2), \ell^\infty)$ has the unique pseudo-expectation property.

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$(B(\ell^2), \ell^\infty)$ has the unique pseudo-expectation property.

Theorem (Pitts '11)

Let $(\mathcal{C}, \mathcal{D})$ be a regular MASA inclusion, i.e., $\mathcal{D} \subseteq \mathcal{C}$ is a MASA and $\overline{span}(N(\mathcal{C}, \mathcal{D})) = \mathcal{C}$. Then $(\mathcal{C}, \mathcal{D})$ has the unique pseudo-expectation property.

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Proposition

Suppose $(\mathcal{C}, \mathcal{D})$ has the unique pseudo-expectation property. If $\mathcal{D} \subseteq \mathcal{C}_1 \subseteq \mathcal{C}$ is a $C^\star$-subalgebra, then $(\mathcal{C}_1, \mathcal{D})$ has the unique pseudo-expectation property.

The Unique Pseudo-Expectation Property + Faithfulness

Proposition

Suppose $(\mathcal{C}, \mathcal{D})$ has the unique pseudo-expectation property and $\mathbb{E} : \mathcal{C} \to I(\mathcal{D})$ is faithful. If $\pi : \mathcal{C} \to B(\mathcal{H})$ is a unital $\star$-homomorphism and $\pi|_{\mathcal{D}}$ is faithful, then π is faithful.

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Proof.

We may assume $\mathcal{D} \subseteq \pi(\mathcal{C})$ and $\pi|_{\mathcal{D}} = \mathrm{id}$. By injectivity, there exists a ucp map $\Phi : \pi(\mathcal{C}) \to I(\mathcal{D})$ such that $\Phi|_{\mathcal{D}} = \mathrm{id}$. Then $\Phi \circ \pi : \mathcal{C} \to I(\mathcal{D})$ is a ucp map such that $(\Phi \circ \pi)|_{\mathcal{D}} = \mathrm{id}$, and so $\Phi \circ \pi = \mathbb{E}$. If $\pi(x) = 0$, then

$$\mathbb{E}(x^\star x) = \Phi(\pi(x^\star x)) = \Phi(\pi(x)^\star \pi(x)) = 0.$$

Since $\mathbb{E}$ is faithful, $x = 0$. □

The Unique Pseudo-Expectation Property + Faithfulness

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Suppose $(\mathcal{C}, \mathcal{D})$ has the unique pseudo-expectation property and $\mathbb{E} : \mathcal{C} \to I(\mathcal{D})$ is faithful. If $\pi : \mathcal{C} \to B(\mathcal{H})$ is a unital $\star$-homomorphism and $\pi|_{\mathcal{D}}$ is faithful, then π is faithful.

Proof.

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Corollary

Suppose $(\mathcal{C}, \mathcal{D})$ has the unique pseudo-expectation property and $\mathbb{E} : \mathcal{C} \to I(\mathcal{D})$ is faithful. If $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$ is a unital operator algebra, then $C^\star_{\text{env}}(\mathcal{A}) = C^\star(\mathcal{A})$.

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Proof.

Since $\mathcal{D}_2$ norms $\mathcal{M}_2$, it norms $\mathcal{A}_2$. By Pitts' Automatic Complete-Boundedness Theorem,

$$\|\theta\|_{cb} \leq \|\theta\|\|\theta^{-1}\|^4 = 1.$$

Likewise, since $\mathcal{D}_1$ norms $\mathcal{M}_1$, it norms $\mathcal{A}_1$, and so

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3. $\theta(GN(\mathcal{A}_1, \mathcal{D}_1)) = GN(\mathcal{A}_2, \mathcal{D}_2)$.

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3. $\theta(GN(\mathcal{A}_1, \mathcal{D}_1)) = GN(\mathcal{A}_2, \mathcal{D}_2)$.

Proof.

Mercer's Representation Theorem. □

Step 2: Replace weak with norm

Define:

- $\mathcal{A}_i^\circ = \overline{span}(GN(\mathcal{A}_i, \mathcal{D}_i))$, a unital operator algebra
- $\mathcal{M}_i^\circ = C^\star(\mathcal{A}_i^\circ)$, a unital $C^\star$-algebra
- $\theta^\circ = \theta|_{\mathcal{A}_1^\circ} : \mathcal{A}_1^\circ \to \theta(\mathcal{A}_1^\circ) \subseteq \mathcal{A}_2$, a completely isometric isomorphism

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Note that:

- $\mathcal{D}_i \subseteq \mathcal{A}_i^\circ \subseteq \mathcal{M}_i^\circ$
- $\mathcal{D}_i \subseteq \mathcal{M}_i^\circ$ is a MASA
- $\overline{span}(GN(\mathcal{M}_i^\circ, \mathcal{D}_i)) = \mathcal{M}_i^\circ$
- $(\mathcal{M}_i^\circ, \mathcal{D}_i)$ has the unique pseudo-expectation property
- The unique ucp map $\mathbb{E}_i : \mathcal{M}_i^\circ \to \mathcal{D}_i$ such that $\mathbb{E}_i|_{\mathcal{D}_i} = \mathrm{id}$ is faithful
- $C^\star_{\mathrm{env}}(\mathcal{A}_i^\circ) = \mathcal{M}_i^\circ$
- $\overline{\mathcal{A}_i^\circ}^\sigma = \mathcal{A}_i$
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In fact:

- $(\mathcal{M}_i^\circ, \mathcal{D}_i)$ is a $C^\star$-diagonal in the sense of Kumjian ('86)

Step 3: Extend θ° to π°

There exists a unique $\star$-isomorphism $\pi^\circ : \mathcal{M}_1^\circ \to \mathcal{M}_2^\circ$ which extends $\theta^\circ : \mathcal{A}_1^\circ \to \mathcal{A}_2^\circ$.

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Proof.

Since $\theta^\circ : \mathcal{A}_1^\circ \to \mathcal{A}_2^\circ$ is a completely isometric isomorphism, there exists a unique $\star$-isomorphism $\pi^\circ : C^\star_{\text{env}}(\mathcal{A}_1^\circ) \to C^\star_{\text{env}}(\mathcal{A}_2^\circ)$ such that $\pi^\circ|_{\mathcal{A}_1^\circ} = \theta^\circ$. But $C^\star_{\text{env}}(\mathcal{A}_i^\circ) = \mathcal{M}_i^\circ$. □

Step 4: Define an implementing unitary for π°

There exists a cyclic and separating vector ξ_1 for $\mathcal{M}_1 \subseteq B(\mathcal{H}_1)$ and a cyclic vector ξ_2 for $\mathcal{M}_2 \subseteq B(\mathcal{H}_2)$ such that

$$\mathcal{M}_1^\circ \xi_1 \to \mathcal{M}_2^\circ \xi_2 : x\xi_1 \mapsto \pi^\circ(x)\xi_2$$

is isometric. Thus there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\pi^\circ(x) = UxU^\star, \; x \in \mathcal{M}_1^\circ .$$

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Proof.

Straightforward but a little tedious. □

Step 5: Conclusion

Define

$$\pi(x) = UxU^\star,\ x \in \mathcal{M}_1 .$$

Then $\pi : \mathcal{M}_1 \to \mathcal{M}_2$ is a σ-weakly continuous $\star$-isomorphism such that $\pi|_{\mathcal{M}_1^\circ} = \pi^\circ$. Since

$$\pi|_{\mathcal{A}_1^\circ} = \pi^\circ|_{\mathcal{A}_1^\circ} = \theta^\circ = \theta|_{\mathcal{A}_1^\circ}$$

and θ is σ-weakly continuous,

$$\pi|_{\mathcal{A}_1} = \theta.$$

Future Directions

1. Rely less on Feldman-Moore. In particular, eliminate the use of Mercer's Representation Theorem.
2. Prove Mercer's Extension Theorem in the norm context. ✓ (Pitts)
3. Study (characterize?) the unique pseudo-expectation property.

Thanks

Thanks for your attention!

Questions?