# Aperiodicity Conditions in Topological $k$-Graphs

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The College of the Holy Cross

2011 Fall Central Sectional Meeting of the AMS

University of Nebraska, Lincoln

Special Session ∼ Recent Progress in Operator Algebras

October 15, 2011

# What's Going On Here?

# What's a Graph Algebra?

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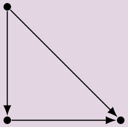

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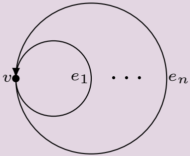


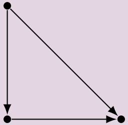

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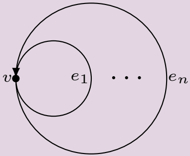


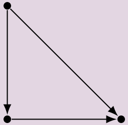

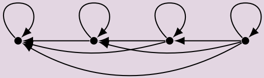

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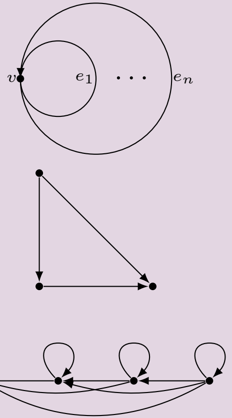


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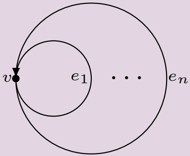


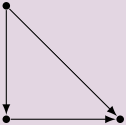

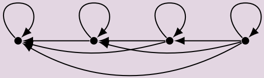

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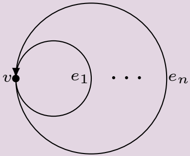


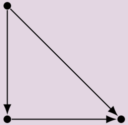

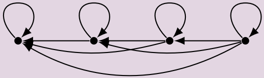

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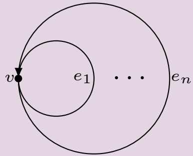


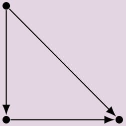

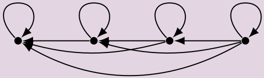

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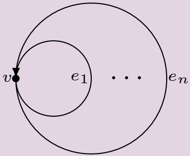


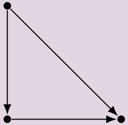

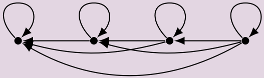

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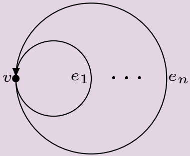


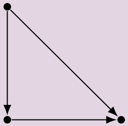

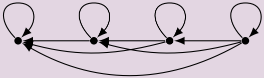

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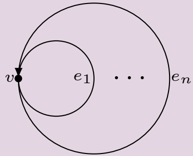


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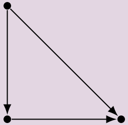

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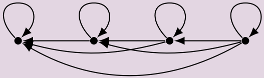

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What do the combinatorics of the graph tell us about the C*-algebra?

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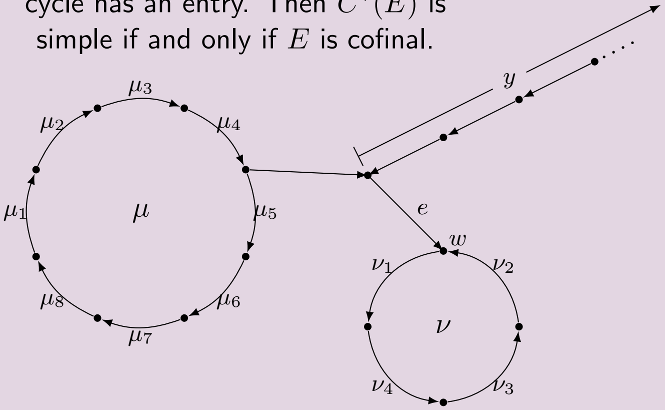

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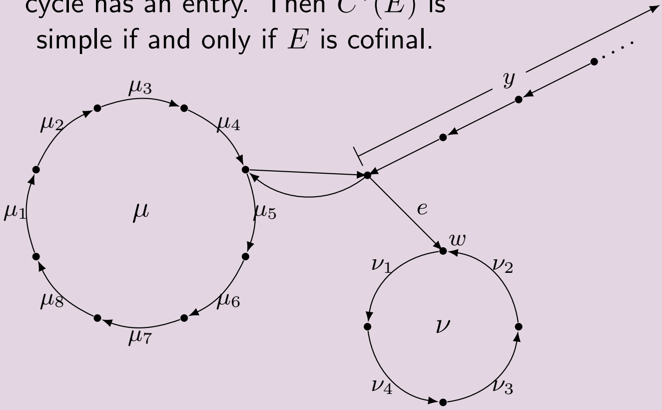

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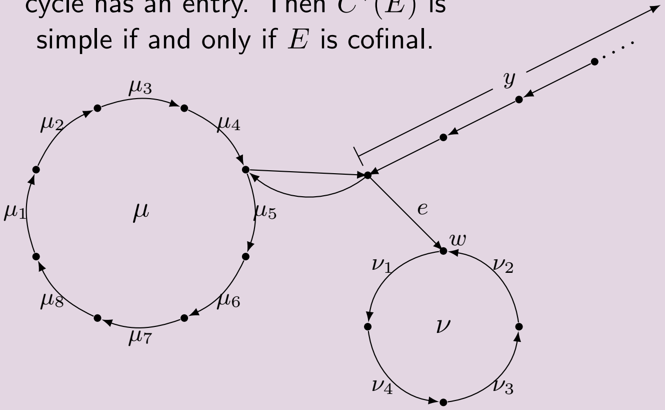

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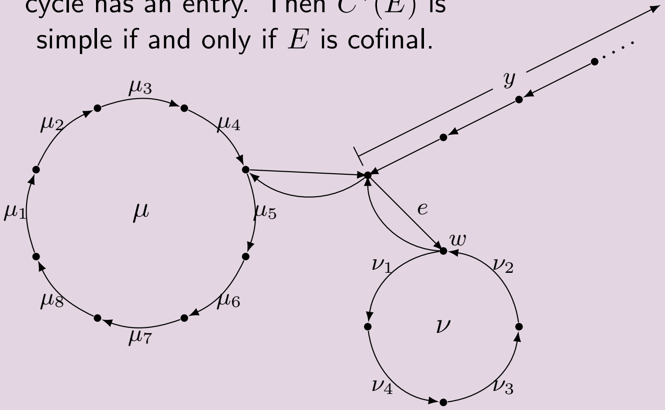

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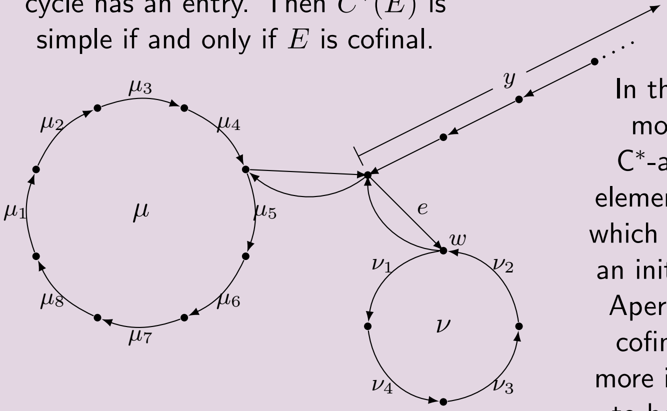


In the groupoid model of the C*-algebra, the elements are paths which differ only by an initial segment. Aperiodicity and cofinality cause more infinite paths to be "related".

# Topological $k$-Graphs

For $k \in \mathbb{N}$, a **topological $k$-graph** is a pair $(\Lambda, d)$ consisting of a category $\Lambda = (\mathrm{Obj}(\Lambda), \mathrm{Mor}(\Lambda), r, s)$ and a functor $d : \Lambda \to \mathbb{N}^k$, called the degree map, which satisfy:

1. $\mathrm{Obj}(\Lambda)$ and $\mathrm{Mor}(\Lambda)$ are second countable, locally compact Hausdorff spaces;
2. $r, s : \mathrm{Mor}(\Lambda) \to \mathrm{Obj}(\Lambda)$ are continuous and $s$ is a local homeomorphism;
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4. $d$ is continuous, where $\mathbb{N}^k$ is given the discrete topology;
5. The unique factorization property: For all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there exists unique $(\mu, \nu) \in \Lambda \times_c \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$.

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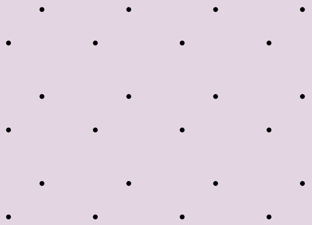

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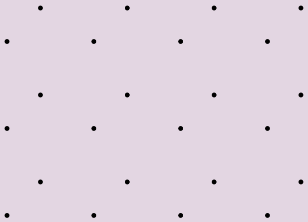

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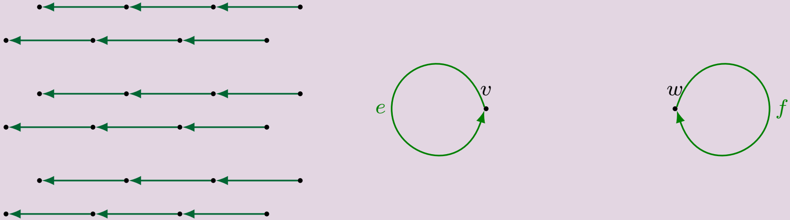

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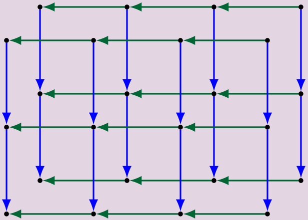

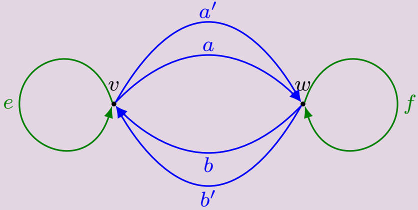

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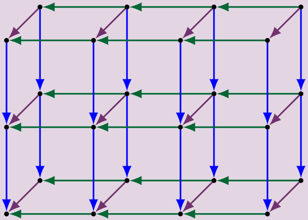

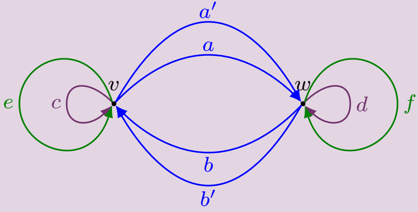

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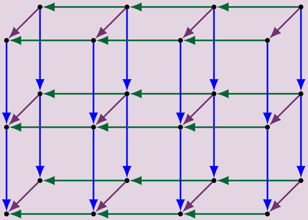

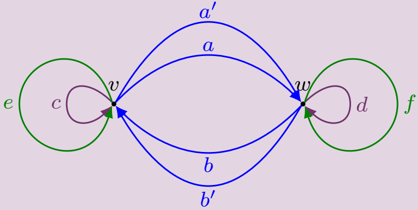


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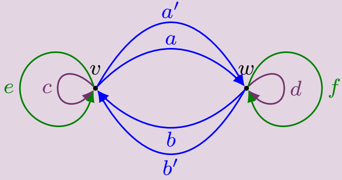
# Infinite Paths

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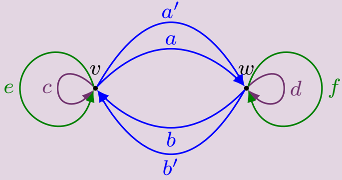


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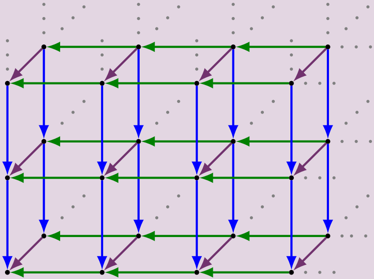

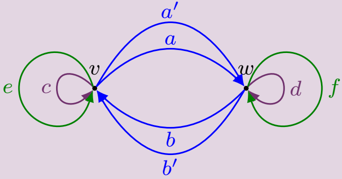


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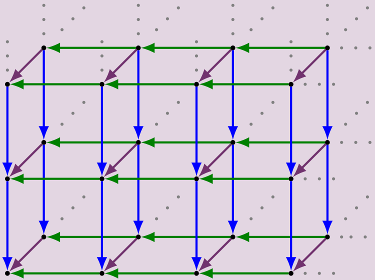

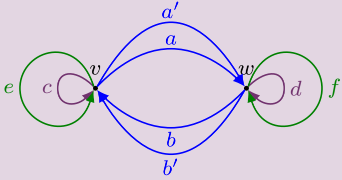


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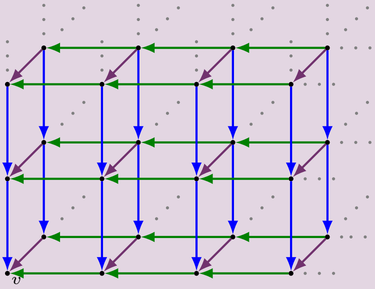


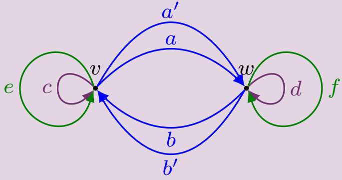


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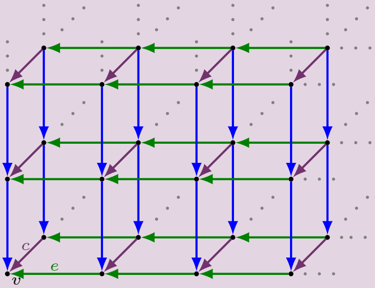


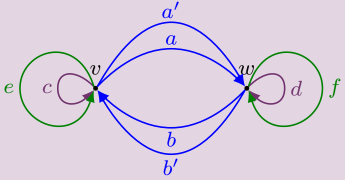


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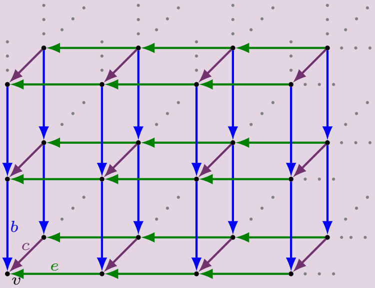


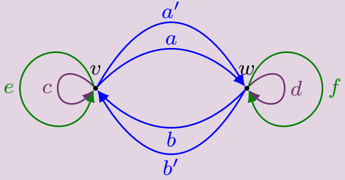


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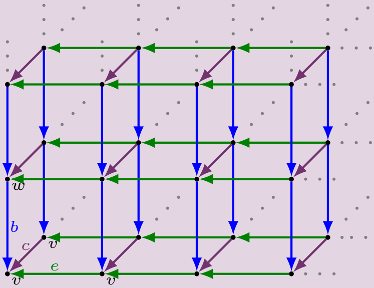


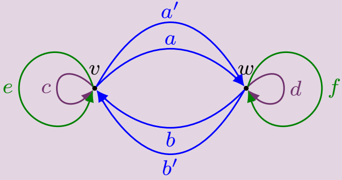


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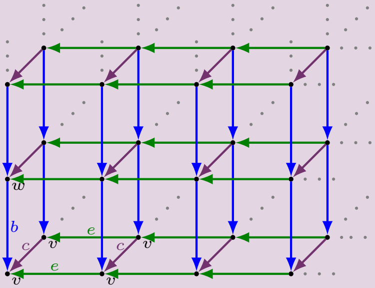


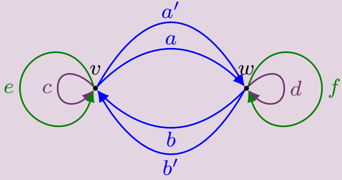


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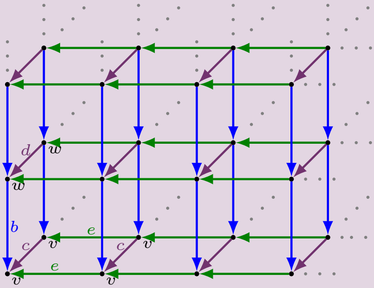


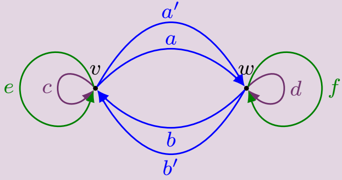


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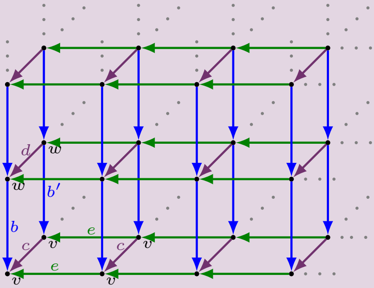


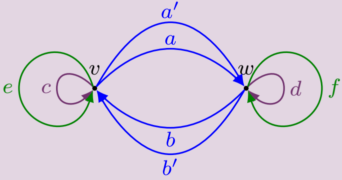


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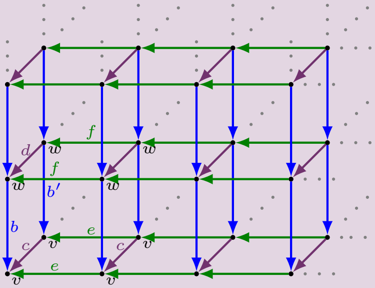


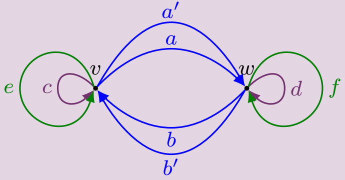


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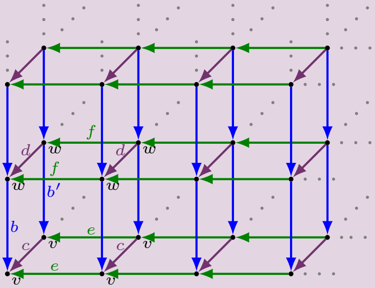


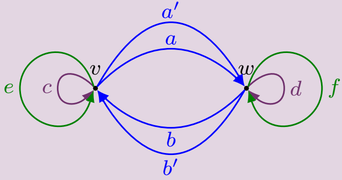


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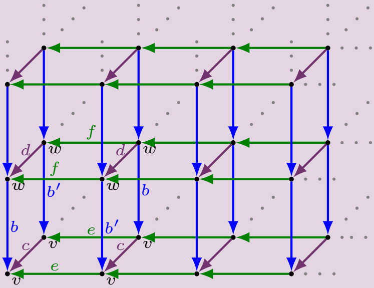


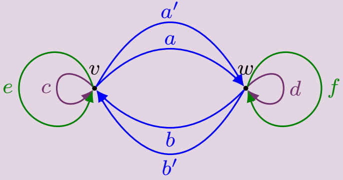


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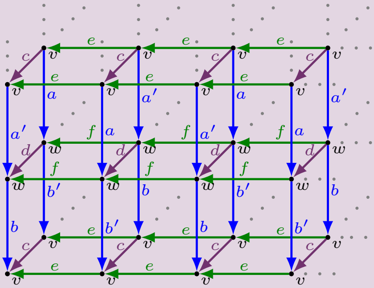


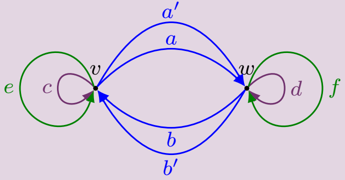


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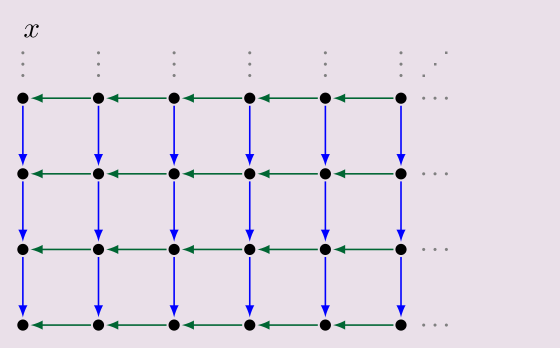

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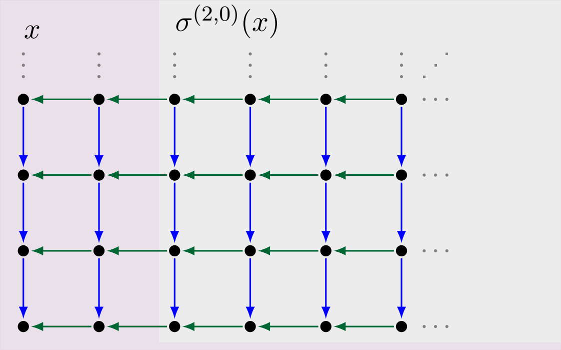

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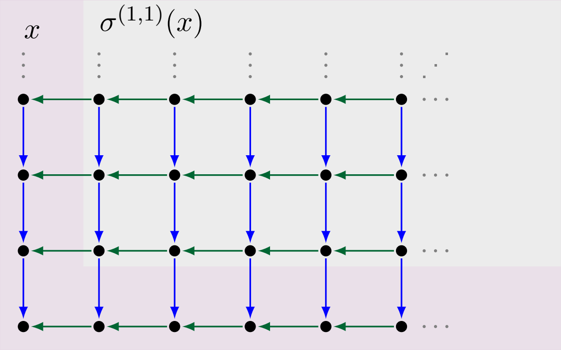

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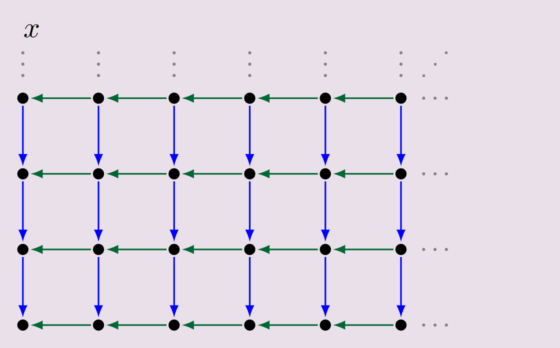


We say that a path $x$ is **aperiodic** if $\sigma^m x = \sigma^n x$ only when $m = n$.

# Yeend's Condition (A)

## Definition

We say a topological $k$-graph $(\Lambda, d)$ is **aperiodic**, or satisfies Condition (A), if for every open set $V \subseteq \Lambda^0$ there exists an infinite aperiodic path $x \in V\Lambda$.

## Proposition (Yeend, 2007)

*Suppose $\Lambda$ is a compactly aligned topological $k$-graph that satisfies Condition (A). Then, $\mathcal{G}_\Lambda$ is topologically principal.*

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(A) There exists an aperiodic path $x \in V\partial\Lambda$.

(B) For any pair $m \neq n \in \mathbb{N}^k$ there exists a path $\lambda_{V,m,n} \in V\Lambda$ such that $d(\lambda) \geq m \vee n$ and

$$\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)). \qquad (\star)$$

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(C) There is a vertex $v \in V$ and paths $\alpha, \beta \in \Lambda$ with $s(\alpha) = s(\beta) = v$ such that there exists a path $\tau \in s(\alpha)\Lambda$ with $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$.

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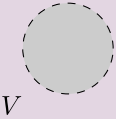

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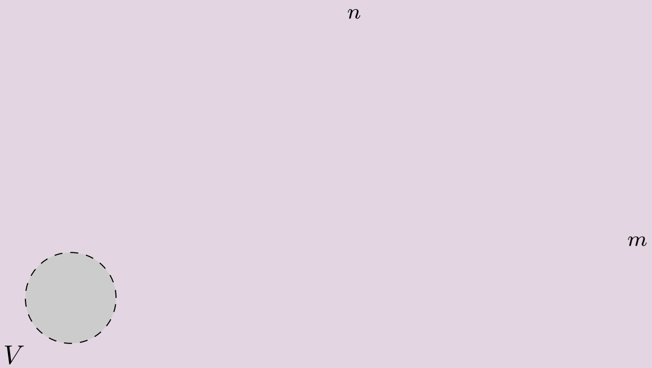

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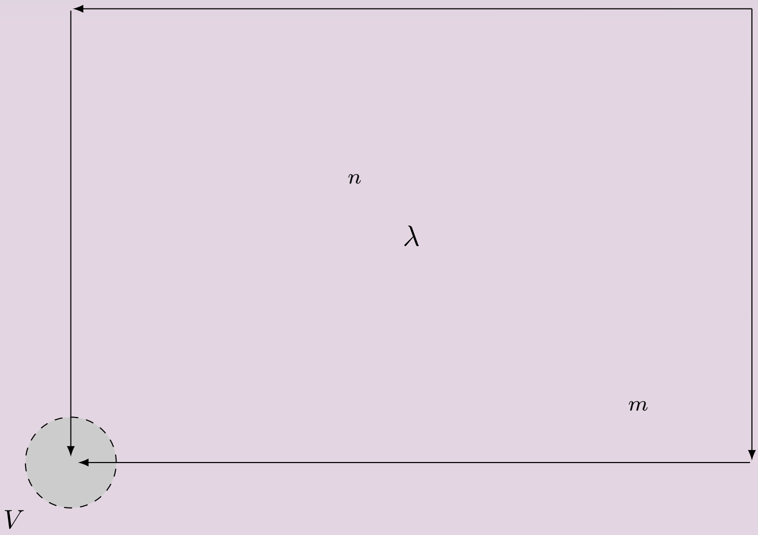

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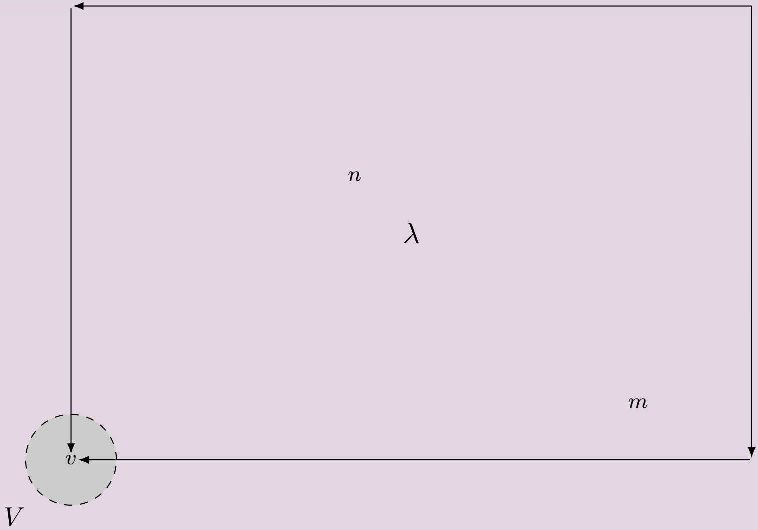

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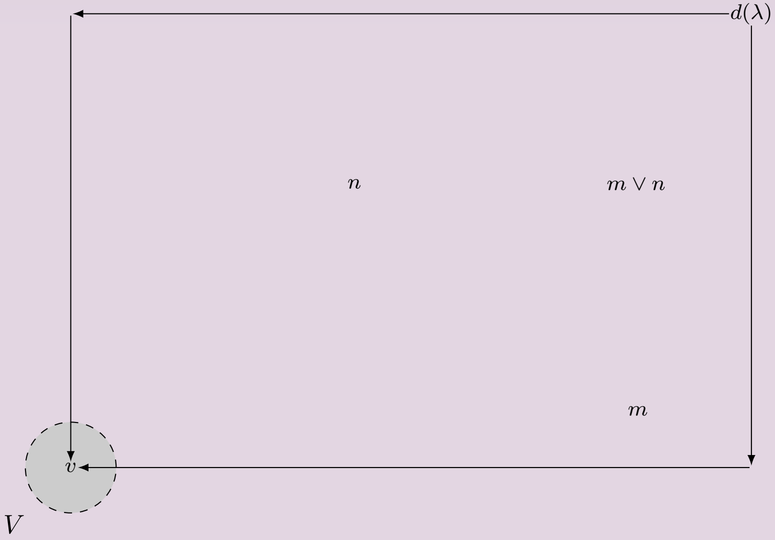

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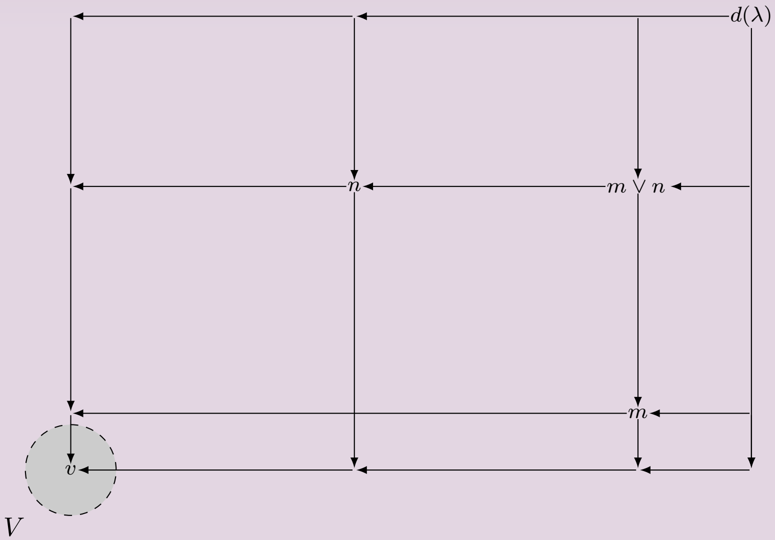

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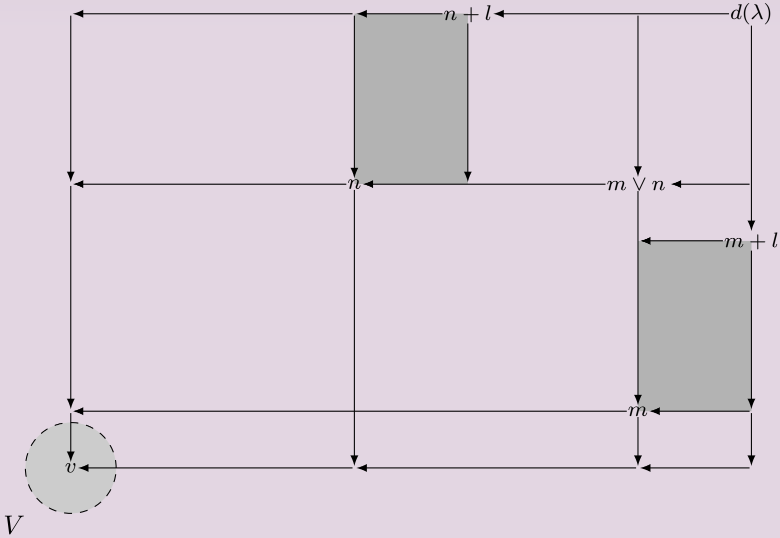

# The Ever Important Lemma

**Tube Lemma**

*Let $V \subset \Lambda^0$ be open, $m \neq n \in \mathbb{N}^k$, and $\lambda \in V\Lambda$ satisfy $(\star)$. Then there exists a compact neighborhood $E \subset V\Lambda^{d(\lambda)}$ of $\lambda$ such that every $\mu \in E$ satisfies $(\star)$.*

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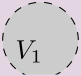

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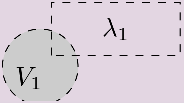

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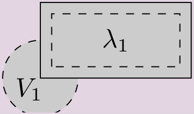

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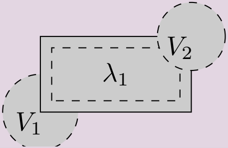

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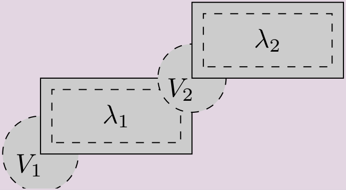

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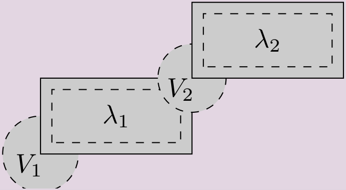

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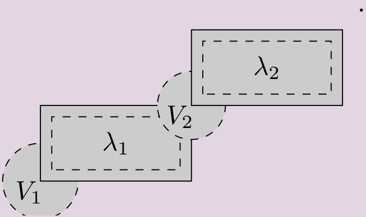

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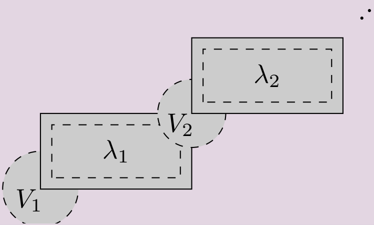

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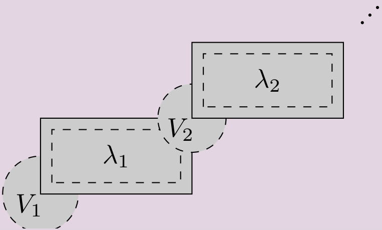

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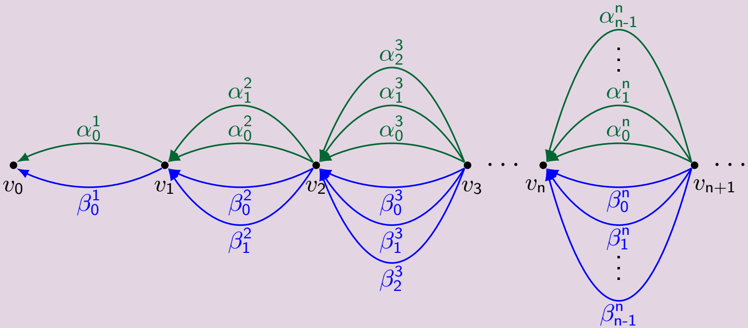

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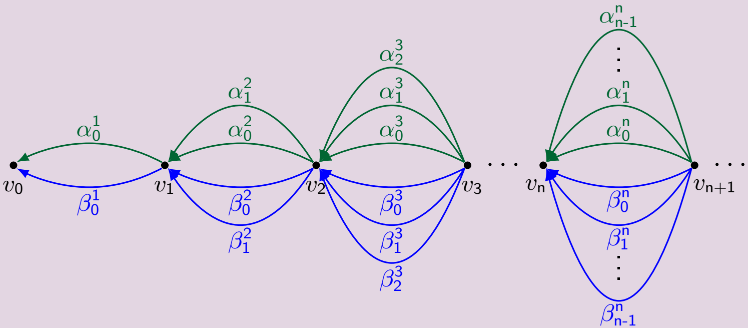


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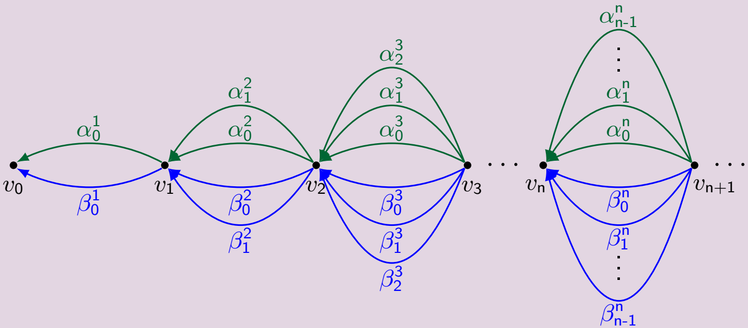


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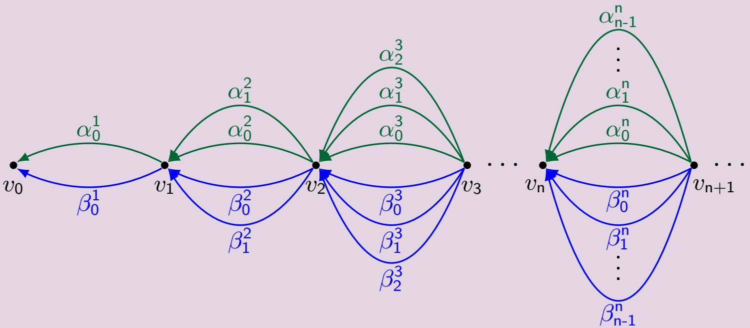


$\alpha_i^n \beta_j^{n+1} = \beta_{i+1}^n \alpha_{j+1}^{n+1}$ $\qquad X := \mathbb{T}$ $\qquad \tau_{\alpha_i^n}(z) = \tau_{\beta_j^n}(z) := z^n$

# Checking Condition (C)

Fix and open set $V \subset \Lambda^0$, $\mu, \nu \in V\Lambda$, with $r(\mu) = r(\nu)$, $s(\mu) = s(\nu)$, and $d(\mu) \wedge d(\nu) = 0$.

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2. Calculate $\sigma^{(1,0)}x$ and $\sigma^{(0,1)}x$.

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7. More complicated 1-skeletons, factorizations, spaces, twistings... EEEKK!

# You're The BEST!

☺ THANKS! ☺

# References


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