Spectral triples for equicontinuous actions and metrics on state spaces

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Joint work with Jean Bellissard and Matilde Marcolli
"Dynamical systems on spectral metric spaces"

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Lemma

$\mathcal{C}^1(X)$ with $\|a\|_D := \|a\| + \|[D, \pi(a)]\|$ is a Banach $*$-algebra, which is closed under holomorphic functional calculus in A.

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The geodesic distance between $x, y \in M$ can be calculated using the above data:

$$d_g(x, y) = \sup\{|f(x) - f(y)|\,;\, f \in A,\ \|[D, \pi(f)]\| \leq 1\}.$$

In fact, $\|[D, \pi(f)]\| = \|\nabla f\|_\infty = \|f\|_{Lip}$. Hence, $\mathcal{C}^1(X) = \mathrm{Lip}(M)$.

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Proposition

If $\lim_{g\to\infty} \ell(g) = \infty$ *then* $(C_r^*(\Gamma), \ell^2(\Gamma), M_\ell)$ *is a spectral triple. Moreover,* $\|[M_\ell, \lambda(g)]\| = \ell(g)$*, for all* $g \in \Gamma$*.*

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Result (Rennie-Varilly)

Let (A, H, D) be a spectral triple such that A is a separable unital C^*-algebra and 1_A acts as the identity operator on H (i.e. the representation is non-degenerate). Assume that the metric commutant $A'_D = \mathbb{C}1_A$. Then d_D is a metric on $S(A)$.

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Caution: The conditions (ii) or (iii) are often very difficult to check! Very different approaches were taken to treat the group C^*-algebras of $\mathbb{Z}^n$ and hyperbolic groups.

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$\hat{u}$ unitary on K and $(\hat{\pi}, \hat{u})$ is a covariant pair $\Rightarrow$ $\hat{\pi}$ extends to $A \rtimes_\alpha \mathbb{Z}$ by $\hat{\pi}(\sum_n a_n u^n) = \sum_n \hat{\pi}(a_n)\hat{u}^n$

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Theorem (Bellissard-Marcolli-R)

$(A \rtimes_\alpha \mathbb{Z}, K, \widehat{D})$ is a spectral triple. Moreover, if (A, H, D) is a spectral metric space and α is an equicontinuous automorphism of A, then $(A \rtimes_\alpha \mathbb{Z}, K, \widehat{D})$ is a spectral metric space.

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- Crossed product $C(M) \rtimes \mathbb{Z}$ of a diffeomorphism ϕ on a compact Riemannian (spin^c) manifold (M, g), with the following condition

$$\sup_{k\in\mathbb{Z}} \left(\sup_{P \neq P' \in M} \frac{d_g\left(\phi^k(P), \phi^k(P')\right)}{d_g(P, P')} \right) < \infty$$

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- Bunce-Deddens algebras: odometer action on the Cantor set