Fuglede-Kadison Determinants for Operators in the Von Neumann Algebra of an Equivalence Relation

Gabriel Picioroaga, University of South Dakota
joint work with Catalin Georgescu, University of South Dakota

Lincoln, October 15, 2011

Fuglede-Kadison determinant

Set-up: (M, τ) II_1-factor, $T \in M$ normal, $|T| = \sqrt{T^*T}$.
$\exists E(\lambda)$ projection-valued measure

$$T = \int_{\sigma(T)} \lambda dE(\lambda)$$

$\mu_T = \tau \circ E \in \mathrm{Prob}(\mathbf{C})$, $\mathrm{supp}(\mu_T) = \sigma(T)$.

Fuglede-Kadison determinant

Set-up: (M, τ) II_1-factor, $T \in M$ normal, $|T| = \sqrt{T^*T}$.
$\exists E(\lambda)$ projection-valued measure

$$T = \int_{\sigma(T)} \lambda dE(\lambda)$$

$\mu_T = \tau \circ E \in \mathrm{Prob}(\mathbf{C})$, $\mathrm{supp}(\mu_T) = \sigma(T)$.

Fuglede-Kadison determinant(1952)

Fuglede-Kadison determinant

Set-up: (M, τ) II_1-factor, $T \in M$ normal, $|T| = \sqrt{T^*T}$.
$\exists E(\lambda)$ projection-valued measure

$$T = \int_{\sigma(T)} \lambda dE(\lambda)$$

$\mu_T = \tau \circ E \in \text{Prob}(\mathbf{C})$, $\text{supp}(\mu_T) = \sigma(T)$.

Fuglede-Kadison determinant(1952)

M finite vN algebra with trace $\tau, T \in M$ invertible

Fuglede-Kadison determinant

Set-up: (M, τ) II_1-factor, $T \in M$ normal, $|T| = \sqrt{T^*T}$.
$\exists E(\lambda)$ projection-valued measure

$$T = \int_{\sigma(T)} \lambda dE(\lambda)$$

$\mu_T = \tau \circ E \in \mathrm{Prob}(\mathbf{C})$, $\mathrm{supp}(\mu_T) = \sigma(T)$.

Fuglede-Kadison determinant(1952)

M finite vN algebra with trace $\tau, T \in M$ invertible

$$\Delta(T) = \exp(\tau(\log|T|))$$

Fuglede-Kadison determinant

Set-up: (M, τ) II_1-factor, $T \in M$ normal, $|T| = \sqrt{T^*T}$.
$\exists E(\lambda)$ projection-valued measure

$$T = \int_{\sigma(T)} \lambda dE(\lambda)$$

$\mu_T = \tau \circ E \in \mathrm{Prob}(\mathbf{C})$, $\mathrm{supp}(\mu_T) = \sigma(T)$.

Fuglede-Kadison determinant(1952)

M finite vN algebra with trace $\tau, T \in M$ invertible

$$\Delta(T) = \exp(\tau(\log|T|))$$

Extend to any $T \in M$:

$$\Delta(T) = \exp\left(\int_0^\infty \log t d\mu_{|T|}\right)$$

Computation Rules

For $n \times n$ matrices $\tau = \mathrm{Tr}/n$: $\Delta(T) = \sqrt[n]{|\det T|}$

Computation Rules

For $n \times n$ matrices $\tau = \mathrm{Tr}/n$: $\Delta(T) = \sqrt[n]{|\det T|}$

$$\Delta(ST) = \Delta(S)\Delta(T)$$

$$\Delta(S) = \Delta(|S|) = \Delta(S^*)$$

$$\Delta(U) = 1 \text{ where } U \text{ is unitary}$$

$$\Delta(\lambda I) = |\lambda|$$

Δ is upper-semicontinuous both in SOT and $\|\cdot\|$

Some recent use of Δ

- [Haagerup & Schultz, 2009] Calculation of Brown measures, solution of the invariant subspace problem for operators in II_1-factors.
- [Bowen, Deninger, H.Li 2006-] Entropy of algebraic actions of discret amenable groups, Ljapunov exponents etc.

Some recent use of Δ

- [Haagerup & Schultz, 2009] Calculation of Brown measures, solution of the invariant subspace problem for operators in II_1-factors.
- [Bowen, Deninger, H.Li 2006-] Entropy of algebraic actions of discret amenable groups, Ljapunov exponents etc.

A computational example

Theorem (Deninger, 2009)

(X, μ) *probability space,* $f \in L^\infty(X)$ *non-zero* μ*-a.e.*
$M = L^\infty(X) \ltimes_\alpha \mathbf{Z}$, $\alpha(1) = g : X \to X$ *ergodic measure preserving.*

Some recent use of Δ

- [Haagerup & Schultz, 2009] Calculation of Brown measures, solution of the invariant subspace problem for operators in II_1-factors.
- [Bowen, Deninger, H.Li 2006-] Entropy of algebraic actions of discret amenable groups, Ljapunov exponents etc.

A computational example

Theorem (Deninger, 2009)

(X, μ) probability space, $f \in L^\infty(X)$ non-zero μ-a.e.
$M = L^\infty(X) \ltimes_\alpha \mathbf{Z}$, $\alpha(1) = g : X \to X$ ergodic measure preserving.
U the unitary implemented by the action :

Some recent use of Δ

- [Haagerup & Schultz, 2009] Calculation of Brown measures, solution of the invariant subspace problem for operators in II_1-factors.
- [Bowen, Deninger, H.Li 2006-] Entropy of algebraic actions of discret amenable groups, Ljapunov exponents etc.

A computational example

Theorem (Deninger, 2009)

(X, μ) probability space, $f \in L^\infty(X)$ non-zero μ-a.e.
$M = L^\infty(X) \ltimes_\alpha \mathbf{Z}$, $\alpha(1) = g : X \to X$ ergodic measure preserving.
U the unitary implemented by the action :

$$\log \Delta(I + fU) = \int_X \log |f(x)| d\mu(x)$$

Some recent use of Δ

- [Haagerup & Schultz, 2009] Calculation of Brown measures, solution of the invariant subspace problem for operators in II_1-factors.
- [Bowen, Deninger, H.Li 2006-] Entropy of algebraic actions of discret amenable groups, Ljapunov exponents etc.

A computational example

Theorem (Deninger, 2009)

(X, μ) *probability space,* $f \in L^\infty(X)$ *non-zero* μ*-a.e.*
$M = L^\infty(X) \ltimes_\alpha \mathbf{Z}$, $\alpha(1) = g : X \to X$ *ergodic measure preserving.*
U *the unitary implemented by the action :*

$$\log \Delta(I + fU) = \int_X \log |f(x)| d\mu(x)$$

Problem: Calculate Δ for other (*ergodic*, measure-preserving) group actions.

Countable Equivalence Relations

Set-up: $(X, \mathcal{B}, \mu)$ probability space, $A_i, B_i \in \mathcal{B}$
$g_i : A_i \to B_i$ measure preserving bijections (Borel partial isomorphisms), $i \in I$ and I countable.

Countable Equivalence Relations

Set-up: $(X, \mathcal{B}, \mu)$ probability space, $A_i, B_i \in \mathcal{B}$
$g_i : A_i \to B_i$ measure preserving bijections (Borel partial isomorphisms), $i \in I$ and I countable.

$(g_i)_i$ generate an equivalence relation $\mathcal{R}$ with countable orbits:

$$x \sim y \text{ if } g_{i_1}^{\varepsilon_1} \dots g_{i_k}^{\varepsilon_k} x = y$$

Countable Equivalence Relations

Set-up: $(X, \mathcal{B}, \mu)$ probability space, $A_i, B_i \in \mathcal{B}$
$g_i : A_i \to B_i$ measure preserving bijections (Borel partial isomorphisms), $i \in I$ and I countable.

$(g_i)_i$ generate an equivalence relation $\mathcal{R}$ with countable orbits:

$$x \sim y \text{ if } g_{i_1}^{\varepsilon_1} \dots g_{i_k}^{\varepsilon_k} x = y$$

Definition

$\mathcal{R}$ is called ergodic if for any measurable set $A \in \mathcal{B}$ with $A = \mathcal{R}(A)$ we have $\mu(A) = 0$ or $\mu(A) = 1$.

Countable Equivalence Relations

Set-up: $(X, \mathcal{B}, \mu)$ probability space, $A_i, B_i \in \mathcal{B}$
$g_i : A_i \to B_i$ measure preserving bijections (Borel partial isomorphisms), $i \in I$ and I countable.

$(g_i)_i$ generate an equivalence relation $\mathcal{R}$ with countable orbits:

$$x \sim y \text{ if } g_{i_1}^{\varepsilon_1} \dots g_{i_k}^{\varepsilon_k} x = y$$

Definition

$\mathcal{R}$ is called ergodic if for any measurable set $A \in \mathcal{B}$ with $A = \mathcal{R}(A)$ we have $\mu(A) = 0$ or $\mu(A) = 1$.

Definition

$\mathcal{R}$ is called treeable if
$\mu\{x \mid \omega(x) = x\} = 0$ for every $\omega = g_1^{\varepsilon_1} g_2^{\varepsilon_2} \dots g_k^{\varepsilon_k}$ *reduced word.*

Example

• If Γ is a countable group then any free, ergodic action on a standard probability space gives rise to an (SP1) equivalence relation on that space.

Example

• If Γ is a countable group then any free, ergodic action on a standard probability space gives rise to an (SP1) equivalence relation on that space.

• There is always an action: any countable group is acting freely and ergodically on $X = \{0,1\}^{\Gamma}$ equipped with the product measure by means of the Bernouli shifts.

Example

• If Γ is a countable group then any free, ergodic action on a standard probability space gives rise to an (SP1) equivalence relation on that space.

• There is always an action: any countable group is acting freely and ergodically on $X = \{0,1\}^{\Gamma}$ equipped with the product measure by means of the Bernouli shifts.

• Countable groups that give rise to treeable, ergodic equivalence relations: free groups, amenable etc.

Example

• If Γ is a countable group then any free, ergodic action on a standard probability space gives rise to an (SP1) equivalence relation on that space.

• There is always an action: any countable group is acting freely and ergodically on $X = \{0, 1\}^\Gamma$ equipped with the product measure by means of the Bernouli shifts.

• Countable groups that give rise to treeable, ergodic equivalence relations: free groups, amenable etc.

• Not necessary that the domains of the generators be all of X: there are groups of non-integer *cost* hence some of their generators must be defined on measurable pieces of X.

Von Neumann algebra of an equivalence relation

Hilbert space $L^2(\mathcal{R})$ consists of those $\varphi : \mathcal{R} \to \mathbf{C}$

$$\int_X \sum_{(z,x)\in\mathcal{R}} |\varphi(x,z)|^2 d\mu(x) < \infty$$

Von Neumann algebra of an equivalence relation

Hilbert space $L^2(\mathcal{R})$ consists of those $\varphi : \mathcal{R} \to \mathbf{C}$

$$\int_X \sum_{(z,x)\in\mathcal{R}} |\varphi(x,z)|^2 d\mu(x) < \infty$$

For ω a reduced word and $f \in L^\infty(X)$ the operators

$$(L_\omega \Psi)(x,y) = \chi_{D_\omega}(x)\Psi(\omega^{-1}x, y), \quad D_\omega \text{ the domain of } \omega^{-1}$$

$$(M_f \Psi)(x,y) = f(x)\Psi(x,y)$$

Von Neumann algebra of an equivalence relation

Hilbert space $L^2(\mathcal{R})$ consists of those $\varphi : \mathcal{R} \to \mathbf{C}$

$$\int_X \sum_{(z,x)\in\mathcal{R}} |\varphi(x,z)|^2 d\mu(x) < \infty$$

For ω a reduced word and $f \in L^\infty(X)$ the operators

$$(L_\omega \Psi)(x,y) = \chi_{D_\omega}(x)\Psi(\omega^{-1}x, y), \quad D_\omega \text{ the domain of } \omega^{-1}$$

$$(M_f \Psi)(x,y) = f(x)\Psi(x,y)$$

(w-closure, linear span of) generate a von Neumann algebra, $\mathcal{M}(\mathcal{R})$ with trace $\tau(T) = \langle T\delta, \delta\rangle$, where δ is the characteristic function of the diagonal of $\mathcal{R}$.

Our goal in this setting: $\log \Delta(\sum_{i=1}^{n} M_{f_i} L_{g_i})$

Our goal in this setting: $\log \Delta(\sum_{i=1}^{n} M_{f_i} L_{g_i})$

Remark

The case $n = 2$ in can be dealt with by following Deninger provided that g_1 and g_2 are full Borel isomorphisms. If $g_1^{-1}g_2$ or $g_2^{-1}g_1$ is ergodic then one can embedd the calculation of the determinant in the hyperfinite II_1-factor generated by the **Z***-action of $g_1^{-1}g_2$, or $g_2^{-1}g_1$ (notice that the ergodicity of an equivalence relation does not guarantee that of a subrelation).*

Proposition (Deninger)

For an operator Φ in a finite von Neumann algebra with a trace $\tau(1) = 1$ the following formula holds:

$$\Delta(z - \Phi) = |z|,$$

if the following two conditions are satisfied:

$$r(\Phi) < |z| \text{ and if } \tau(\Phi^n) = 0.$$

Lemma

Let $g : A \to B$ a Borel partial isomorphism and denote by fg^{-1} the function $(fg^{-1})(x) = f(g^{-1}x)$. Then we have:

$$L_g M_f = M_{fg^{-1}} L_g$$

$$(M_f L_g)^n = M_{f \cdot fg^{-1} \cdot \ldots \cdot fg^{-(n-1)}} L_{g^n}$$

Lemma

Let $g : A \to B$ a Borel partial isomorphism and denote by fg^{-1} the function $(fg^{-1})(x) = f(g^{-1}x)$. Then we have:

$$L_g M_f = M_{fg^{-1}} L_g$$

$$(M_f L_g)^n = M_{f \cdot fg^{-1} \cdot \ldots \cdot fg^{-(n-1)}} L_{g^n}$$

Proposition

Let $\Phi = \sum_{i=1}^{n} M_{f_i} L_{g_i}$ in $\mathcal{M}(\mathcal{R})$ where $\mathcal{R}$ is ergodic and treeable and the g_i's are Borel partial isomorphisms and among its generators. Then:

$$\tau(\Phi^n) = 0, \quad \forall n \geqslant 1$$

Lemma

Let $g : A \to B$ a Borel partial isomorphism and denote by fg^{-1} the function $(fg^{-1})(x) = f(g^{-1}x)$. Then we have:

$$L_g M_f = M_{fg^{-1}} L_g$$

$$(M_f L_g)^n = M_{f \cdot fg^{-1} \cdot \ldots \cdot fg^{-(n-1)}} L_{g^n}$$

Proposition

Let $\Phi = \sum_{i=1}^{n} M_{f_i} L_{g_i}$ in $\mathcal{M}(\mathcal{R})$ where $\mathcal{R}$ is ergodic and treeable and the g_i's are Borel partial isomorphisms and among its generators. Then:

$$\tau(\Phi^n) = 0, \quad \forall n \geqslant 1$$

Proof based on: $\tau(M_h L_\omega) = \int_{\{x | \omega(x) = x\}} h(x) d\mu(x) = 0$

Theorem

Let $f_i \in L^\infty(X)$ and $T = \sum_{i=1}^n M_{f_i} L_{g_i} \in \mathcal{M}(\mathcal{R})$ such that $g_i : A_i \to B_i$ are Borel partial isomorphisms among the generators of a treeable (SP1) equivalence relation $\mathcal{R}$. Assume that there is an index i_0 such that

$$\sum_{i \neq i_0} ||f_i / f_{i_0}||_\infty < 1,$$

that f_{i_0} is non-vanishing on sets of positive measure and that $g_{i_0} : A_{i_0} = X \to B_{i_0} = X$. Then:

$$\log \Delta(T) = \int_X \log |f_{i_0}| d\mu(x)$$

If $\mathcal{R}$ not necessarilty treeable and all L_{g_i} partial isometries:

If $\mathcal{R}$ not necessarilty treeable and all L_{g_i} partial isometries:

Theorem

Let $f_i \in L^\infty(X)$ and $g_i : A_i \to B_i$, $i = 1, .., n$ be Borel partial isomorphisms in the standard probability space $(X, \mathcal{B}, \mu)$ such that g_i's are among the generators of an ergodic (SP1) equivalence relation.

If $\mathcal{R}$ not necessarilty treeable and all L_{g_i} partial isometries:

Theorem

Let $f_i \in L^\infty(X)$ and $g_i : A_i \to B_i$, $i = 1, .., n$ be Borel partial isomorphisms in the standard probability space $(X, \mathcal{B}, \mu)$ such that g_i's are among the generators of an ergodic (SP1) equivalence relation. If the following conditions are satisfied

- $\mu(A_1 \cup A_2 \ldots \cup A_n) = 1$

If $\mathcal{R}$ not necessarilty treeable and all L_{g_i} partial isometries:

Theorem

Let $f_i \in L^\infty(X)$ and $g_i : A_i \to B_i$, $i = 1, .., n$ be Borel partial isomorphisms in the standard probability space $(X, \mathcal{B}, \mu)$ such that g_i's are among the generators of an ergodic (SP1) equivalence relation. If the following conditions are satisfied

- $\mu(A_1 \cup A_2 \ldots \cup A_n) = 1$
- $\mu(B_i \cap B_j) = 0$ *if* $i \neq j$

If $\mathcal{R}$ not necessarilty treeable and all L_{g_i} partial isometries:

Theorem

Let $f_i \in L^\infty(X)$ and $g_i : A_i \to B_i$, $i = 1, .., n$ be Borel partial isomorphisms in the standard probability space $(X, \mathcal{B}, \mu)$ such that g_i's are among the generators of an ergodic (SP1) equivalence relation. If the following conditions are satisfied

- $\mu(A_1 \cup A_2 \ldots \cup A_n) = 1$
- $\mu(B_i \cap B_j) = 0$ *if* $i \neq j$

then the Kadison-Fuglede determinant of the operator $T = \sum_{i=1}^n M_{f_i} L_{g_i}$ is given by

$$\log \Delta(T) = \sum_{i=1}^{n} \int_{B_i} \log |f_i| d\mu(x)$$

Idea of proof: $L_{g_j}^* L_{g_i} = 0$ and $T^* T$ is a multiplication operator.

$$T^* T = M_{\sum_i |f|^2 g_i \cdot \chi_{A_i}}$$

$$\log \Delta(M_f) = \int_X \log |f| d\mu$$