C*-algebras and Kakutani Equivalence of minimal Cantor systems

A Characterization for minimal $\mathbb{Z}^d$ actions on the Cantor set

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Object of the talk

This talk presents a new form of equivalence between minimal free Cantor systems which has a natural dynamical and a natural C*-algebraic picture.

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This talk is based upon:

Paper

C^*-algebraic characterization of bounded orbit injection equivalence for minimal free Cantor systems

Frédéric Latrémolière, Nic Ormes, **2011**, Rocky Mountain Journal of Mathematics (Accepted in 2009), ArXiv: 0903.1881.

Classifying dynamics

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$$\forall z \in \mathbb{Z}^d \;\; \psi^z \circ h = h \circ \phi^z .$$

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However, there is no good general invariant known, and it is usually a delicate problem.
As the problem of classification up to conjugacy is complicated, one may try to define weaker equivalence notions which may be more approachable. An example is *(strong) Orbit equivalence*.

Kakutani Equivalence: Derived Systems

Let $(X, \phi, \mathbb{Z})$ be a dynamical system and assume that the orbit of each point is dense.

Definition

Let $A \subseteq X$ be a nonempty clopen set. The derived system $(A, \mathbb{Z}, \rho)$ is defined by setting $\rho^1(x)$ to be the first return time of $x \in A$ to A:

$$\forall x \in A \;\; \rho^1(x) = \inf\{n \in \mathbb{N}, n > 0 : \phi^n(x) \in A\}.$$

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Two dynamical systems $(X, \phi, \mathbb{Z})$ and $(Y, \psi, \mathbb{Z})$ are Kakutani equivalent when they are conjugate to derived systems of some dynamical system $(Z, \rho, \mathbb{Z})$.

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How to we generalize this notion to $\mathbb{Z}^d$?

Bounded orbit injection

We define a generalized notion of derived systems for $\mathbb{Z}^d$ actions.

Definition (Lightwood, Ormes 2007)

Let $(X, \phi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ be two dynamical systems. A map $\theta : X \to Y$ is a *orbit injection* when it is a continuous open injection such that for all $x, y \in X$:

$$\exists z \in \mathbb{Z}^d \ \phi^z(x) = y \iff \exists n(z,x) \in \mathbb{Z}^d \ \psi^{n(z,x)}(\theta(x)) = \theta(y).$$

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The idea behind this definition is that a derived system defines a natural orbit injection, and conversely if an orbit injection exists between $\mathbb{Z}$-actions, we have in fact a derived system. The map n is unique if the action ψ is free. An orbit injection is *bounded* when it has a bounded cocycle.

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Definition (Lightwood, Ormes 2007)

Two dynamical systems $(X, \phi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ are bounded orbit injection equivalent when there exists a dynamical system $(Z, \rho, \mathbb{Z}^d)$ and bounded orbit injections from $(X, \phi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ into $(Z, \rho, \mathbb{Z}^d)$.

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This notion generalizes flip-Kakutani equivalence for actions of $\mathbb{Z}^d$.

Are they any good invariant for this relation? For this, we shall restrict ourselves to a class of dynamical systems for which many of our equivalence relations have been successfuly understood.

Minimal Free Cantor Systems

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A cantor set is a topological space homeomorphic to the usual middle-third Cantor set, or equivalently it is a perfect, completely disconnected compact metrizable space.

Definition

Let X be a Cantor set. A minimal free Cantor system is a free action of $\mathbb{Z}^d$ on X by homeomorphisms such that every point has a dense orbit in X.

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Definition

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We will denote a free minimal dynamical system by $(X, \phi, \mathbb{Z}^d)$ where the homeomorphism of X defined by $z \in \mathbb{Z}^d$ is denoted ϕ^z.

Definition

Let $(X, \phi, \mathbb{Z}^d)$ be a dynamical system. The continuous functions on the orbit space is:

$$\{f \in C(X) : \forall z \in \mathbb{Z}^d \;\; f \circ \phi^z = f\}.$$

This reduces to constants when ϕ is minimal. Is there a good (noncommutative) replacement? The idea is to require only that $f \in C(X)$ and $f \circ \phi^z$ be "equivalent" in some way. We arrive at:

Definition

Let $(X, \phi, \mathbb{Z}^d)$ be a dynamical system. The C*-crossed-product $C(X) \rtimes_\phi \mathbb{Z}^d$ is the universal C*-algebra generated by $C(X)$ and unitaries U^z $(z \in \mathbb{Z}^d)$ such that $U^{z+z'} = U^z U^{z'}$ and $U^z f U^{-z} = f \circ \phi^{-z}$ for all $z, z' \in \mathbb{Z}^d$.

This notion was introduced by Zeller-Meier in 68 and has been a major source of examples.

Giordano, Putnam, Skau

The two main results in the subject of minimal free Cantor systems were established by these three authors in 95.

Theorem (GPS, 95)

Two free minimal Cantor systems $(X, \phi, \mathbb{Z})$ and $(Y, \psi, \mathbb{Z})$ are strongly orbit equivalent if and only if their C*-crossed-product algebras are *-isomorphic.

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These C*-algebras are fully characterized by their ordered K-theory, so ordered K-theory is a complete invariant for strong orbit equivalence for minimal free Cantor systems.

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Theorem (GPS, 95)

Two free minimal Cantor systems $(X,\phi,\mathbb{Z})$ and $(Y,\psi,\mathbb{Z})$ are flip conjugate if and only if there is a *-isomorphism μ between their C*-crossed-products mapping $C(X)$ onto $C(Y)$.

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Theorem (GPS, 95)

Two free minimal Cantor systems $(X, \phi, \mathbb{Z})$ and $(Y, \psi, \mathbb{Z})$ are strongly orbit Kakutani equivalent if and only if their C*-crossed-product algebras are Morita equivalent.

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We see that in principle, it is easier to show C*-algebras are Morita equivalent, so strong orbit Kakutani is indeed weaker than conjugacy.
Can we extend this result to characterize flip-Kakutani and its generalization, BOIE?
Note: continuity of the cocycle in orbit equivalence is the key difference between strong orbit eq and bounded orbit eq, the latter being flip-conjugacy in our case.

Morita Equivalence

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Question

What is the appropriate notion of Morita equivalence for C*-algebras?

Though rings, C*-algebras are more. Rieffel proposes a stronger version of equivalence.

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Though rings, C*-algebras are more. Rieffel proposes a stronger version of equivalence.

Definition (Rieffel)

Let A and B be two C*-algebras. They are Rieffel-Morita equivalent when their categories of hermitian modules are equivalent.

Morita Equivalence, 2

We have:

Theorem (Rieffel)

Two C*-algebras A and B are Rieffel-Morita equivalent when there exists two Hilbert C*-bimodule M and N such that $M \otimes_B N = A$ and $N \otimes_A M = B$.

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We have the following important:

Theorem

Let A be a simple C-algebra. Let $p \in A$ be a projection. Then pAp and A are Rieffel-Morita equivalent.*

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Proof.

Let $M = pA$ and $N = Ap$. Note that $ApA = A$ by simplicity. □

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1. $(X, \phi, \mathbb{Z}^d)$ and $(Y, \psi, \mathbb{Z}^d)$ are bounded orbit injection equivalent,
2. There exists a *-monomorphism $\alpha : C(X) \rtimes_\phi \mathbb{Z}^d \to C(Y) \rtimes_\psi \mathbb{Z}^d$ such that the range of α is $pC(Y) \rtimes_\psi \mathbb{Z}^d p$ with $\alpha(1) = p$.

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Thus, BOIE is a stronger notion that Morita equivalence, as we need to recall the base algebra on which the action occurs.

Step 1

To prove our result, we start by defining some important projections:

$$p_h^z(y) = \begin{cases} 1 \text{ if } \; y = \theta(x), x = \phi^z(x') \text{ and } y = \psi^h(\theta(x')) \\ 0 \quad \text{otherwise.} \end{cases}$$

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5. $p_h^{z+z'} = \sum_{h'} p_{h'}^z \cdot p_{h-h'}^{z'} \circ \psi^{-h'}$.

Step 1 - proof outline

$\theta(X)$ is clopen in Y.

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p_h^z and $p_{h'}^z$ are orthogonal.

1. θ is open and continuous.
2. p_h^z is the indicator of the image by θ of $(n(\cdot, z))^{-1}(h)$, which is clopen.
3. Let y such that $p_h^z(y) = p_{h'}^z(y) = 1$. Then: $y = \theta(x)$ and:

$$\psi^h(y) = \theta(\phi^z(x))$$

and same with h'. Since ψ is free we have $h = h'$.

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Let $y \in Y$ and $z, z', h \in \mathbb{Z}^d$ such that $p_h^{z+z'}(y) = 1$. Then there exists $x, x' \in X$ such that:

$$y = \theta(x), x = \phi^{z+z'}(x'), \psi^h(y) = \theta(x').$$

Thus:

$$x = \phi^z(\phi^{z'}(x')).$$

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so there exists $n(\phi^{z'}(x'), z)' := h'$ such that:

$$y = \psi^{h'}(\theta(\phi^{z'}(x')))$$

i.e. $p_{h'}^z(y) = 1$. This is the only possible nonzero $p_?^z(y)$ by orthogonality.

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We had $y = \psi^{h'}(\theta(\phi^{z'}(x')))$. So:

$$\psi^{-h'}(y) = \theta(\phi^{z'}(x')).$$

On the other hand:

$$\psi^{h}(y) = \theta(x')$$

so

$$\psi^{-h'}(y) = \psi^{h-h'}(\theta(x'))$$

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$$p_{h-h'}^{z'}(\psi^{-h'}(y)) = 1.$$

Step 2

We can define unitaries by:

$$V^z = \sum_h p_h^z U_\psi^h + (1-p)$$

and we check that $z \mapsto V^z$ is a group homomorphism into the unitary group of $C(Y) \rtimes_\psi \mathbb{Z}^d$. This uses our convolution formula for the p_h^z.

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We then define $\pi : C(X) \mapsto C(Y)$ by $\pi(f)(y) = f(x)$ if $y = \theta(x)$ and 0 otherwise. π goes backward! However, we check that (π, V) is a covariant representation for $(C(X), \phi, \mathbb{Z}^d)$. To check this is tricky and uses the orbit injection property a lot!

Step 3

1. The pair (π, V) is covariant, so we can define a *-morphism $\alpha : C(X) \rtimes_\phi \mathbb{Z}^d \to C(Y) \rtimes_\psi \mathbb{Z}^d$ which extends it.

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3. We write $pU_\psi^h p$ as a sum of elements all in the range of α by decomposing $\theta(X)$ as the disjoint union of $\{x \in \theta(X) : \psi^h(x) \in \theta(X)\}$.

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Let v be a unitary in $C(X) \rtimes_\phi \mathbb{Z}^d$ such that $vC(X)v = C(X)$. Then:*

$$v = f \sum_z p_z U_\phi^z$$

where $f \in C(X)$ and p_z are mutually ortogonal projections, only finitely nonzero, summing to 1.

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Define $p_z = |\mathbb{E}(vU_\phi^{-z})|$. Consider the regular representation π for the Dirac measure at some $x \in X$, which is faithful (minimality) and irreducible and acts on $l^2(\mathbb{Z}^d)$ by:

$$\pi(U_p^z)\delta_h = \delta_{h+z} \text{ and } \pi(f)(\delta_h) = f(\phi^{-h}(x)).$$

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where $f \in C(X)$ and p_z are mutually ortogonal projections, only finitely nonzero, summing to 1.

Note that $\pi(v)$ commutes with $\pi(C(X))$ so it commutes with $l^\infty(G)$. Hence, as it is a unitary, it is of the form $\pi(v)(\delta_h) = \lambda_h \delta_{\sigma(h)}$ for some permutation σ of $\mathbb{Z}^d$.

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Now, we suppose given a *-monomorphism with the listed properties. Is it induced by some bounded orbit injection?

Lemma (generalized from GPS95)

Let v *be a unitary in* $C(X) \rtimes_\phi \mathbb{Z}^d$ *such that* $vC(X)v* = C(X)$*. Then:*

$$v = f \sum_z p_z U_\phi^z$$

where $f \in C(X)$ *and* p_z *are mutually ortogonal projections, only finitely nonzero, summing to 1.*

A direct computation shows that $\pi(p_z)$ is a projection; that they are mutually orthogonal, and satisfy the desired properties.

Step 5

1. Thus, given a *-monomorphism with the described properties, the image of U_ϕ^z is a unitary V^z which stabilizes $C(Y)$. The projections given by our lemma play the role of p_h^z in the first direction of the proof.

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3. One direction of the orbit injection property — namely that if two points are in the same orbit in Y and images of points in X by θ, the latter are in the same orbit for ϕ — is quite tricky.