Combinatorial Algebra for Normed Structures

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Traditionally [1, 3, 7], universal C*-algebras have been constructed by building a complex *-algebra on a set S subject to certain *-algebraic relations R, and then norming by certain representation restrictions.

$$\begin{array}{ccc} \mathbb{C}^*\langle S\rangle & \twoheadrightarrow & \mathbb{C}^*\langle S|R\rangle \\ & & \downarrow \\ & & \langle S|R\rangle_{\mathbf{1C}^*} \end{array}$$

However, this is counter to the algebraic means of quotienting a free object.

Proposition (Folklore)

Let $S \neq \emptyset$ and $\mathscr{C}$ any subcategory of normed vector spaces with contractive maps. If $\mathrm{Ob}(\mathscr{C})$ *contains $V \ncong \mathbb{O}$, then S has no associated free object in $\mathscr{C}$.*

The classical notions can be recovered by replacing **Set**.

Definition ([3, 4])

A *normed set* is a pair (S, f), where S is a set and f a function from S to $[0, \infty)$.
Given two normed sets (S, f) and (T, g), a function $\phi : S \to T$ is *contractive* if $g(\phi(s)) \leq f(s)$ for all $s \in S$.

Given a normed set (S, f),

1. Form the set $S_f := S \setminus f^{-1}(0)$.
2. Construct the free unital *-algebra $A_{S,f}$ over $\mathbb{C}$ on S_f.
3. Construct a C*-norm on $A_{S,f}$ from f.
4. Complete $A_{S,f}$ into a unital C*-algebra $\mathcal{A}_{S,f}$.

Theorem (Scaled-Free Mapping Property, [6])

Let (S, f) be a normed set, $\mathcal{B}$ a unital C-algebra, and $\phi : (S, f) \to \mathcal{B}$ a function. Then, there is a unique unital *-homomorphism $\hat{\phi} : \mathcal{A}_{S,f} \to \mathcal{B}$ such that*

$$\|\phi(s)\|_{\mathcal{B}} \cdot \hat{\phi}(s) = f(s) \cdot \phi(s)$$

for all $s \in S$.

Definition

A *C*-relation* on (S, f) is an element of $\mathcal{A}_{S,f}$.

Definition

For a crutched set (S, f) and C*-relations $R \subseteq \mathcal{A}_{S,f}$ on (S, f), let J_R be the two-sided, norm-closed ideal generated by R in $\mathcal{A}_{S,f}$. Then, the *unital C*-algebra presented on* (S, f) *subject to* R is

$$\langle S, f | R \rangle_{\mathbf{1C^*}} := \mathcal{A}_{S,f} / J_R.$$

$$\begin{array}{ccc} \mathbb{C}^*\langle S \rangle & \twoheadrightarrow & \mathbb{C}^*\langle S | R \rangle \\ \downarrow & & \downarrow \\ \langle S, f | \emptyset \rangle_{\mathbf{1C^*}} & \twoheadrightarrow & \langle S, f | R \rangle_{\mathbf{1C^*}} \end{array}$$

Fact: This square commutes for all *-algebraic relations R.

For group theory, Tietze ([9], 1908) described canonical means of converting between presentations of the same group.
These same transformations exist for this presentation theory for $\mathbf{1C}^*$.

1. Adding/Removing C*-relations.
   (e.g. $\left\langle (x,\lambda) \,\middle|\, x = x^2 \right\rangle_{\mathbf{1C}^*} \leftrightarrow \left\langle (x,\lambda) \,\middle|\, x = x^2, x = x^5 \right\rangle_{\mathbf{1C}^*}$)
2. Adding/Removing generators.
   (e.g.
   $\left\langle (x,\lambda) \,\middle|\, x = x^2 \right\rangle_{\mathbf{1C}^*} \leftrightarrow \left\langle (x,\lambda), (y,\lambda^2) \,\middle|\, x = x^2, y = x^*x \right\rangle_{\mathbf{1C}^*}$)

One of these transformations is *elementary* if only one generator or C*-relation is altered.

Consider the C*-algebra of a left-invertible element.

$$\mathcal{L} := \left\langle (x, \lambda) \,\middle|\, \mu^2 x^* x \geq \mathbb{1} \right\rangle_{\mathbf{1C^*}} .$$

If $\lambda\mu < 1$, $\|\mathbb{1}\|_{\mathcal{L}} < 1$. Thus, $\mathbb{1} = 0$ so $\mathcal{L} \cong_{\mathbf{1C^*}} \mathbb{O}$.

For $\lambda\mu \geq 1$,

$$\mathcal{L} \cong_{\mathbf{1C^*}} \left\langle \begin{matrix} (x,\lambda),(q,\lambda), \\ (u,\lambda\mu) \end{matrix} \middle| \begin{matrix} \mu^2 x^* x \geq \mathbb{1}, q = (x^*x)^{\frac{1}{2}}, \\ u = \mu x \left(p\left(\mu (x^*x)^{\frac{1}{2}} - \mathbb{1} \right) + \mathbb{1} \right)^{-1} \end{matrix} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle \begin{matrix} (x,\lambda),(q,\lambda), \\ (u,\lambda\mu) \end{matrix} \middle| \begin{matrix} \mu^2 x^* x \geq \mathbb{1}, q = (x^*x)^{\frac{1}{2}}, \\ u = \mu x \left(p\left(\mu (x^*x)^{\frac{1}{2}} - \mathbb{1} \right) + \mathbb{1} \right)^{-1}, \\ \mathbb{1} \leq \mu q, u^*u = \mathbb{1}, x = uq \end{matrix} \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \left\langle \begin{matrix} (x,\lambda),(q,\lambda), \\ (u,\lambda\mu) \end{matrix} \middle| \mathbb{1} \leq \mu q, u^*u = \mathbb{1}, x = uq \right\rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \langle (q,\lambda),(u,\lambda\mu) \,|\, \mathbb{1} \leq \mu q, u^*u = \mathbb{1} \rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} \langle (q,\lambda) | \mathbb{1} \leq \mu q \rangle_{\mathbf{1C^*}} *_{\mathbb{C}} \langle (u,\lambda\mu) \,|\, u^*u = \mathbb{1} \rangle_{\mathbf{1C^*}}$$

$$\cong_{\mathbf{1C^*}} C\left[\frac{1}{\mu}, \lambda\right] *_{\mathbb{C}} \mathcal{T}$$

Consider the C*-algebra of a single idempotent.

$$\mathcal{A} := \left\langle (x,\lambda) \,\middle|\, x = x^2 \right\rangle_{\mathbf{1C^*}} .$$

If $\lambda < 1$, then $x = 0$. Hence, $\mathcal{A} \cong_{\mathbf{1C^*}} \mathbb{C}$.
For $\lambda \geq 1$, the range and kernel projections can be formed from x, [2, Proposition IV.1.1]. Likewise, x can be written in terms of these projections, [10, Theorem 1].

$$\mathcal{A} \quad \cong_{\mathbf{1C^*}} \quad \left\langle (r,1),(k,1) \,\middle|\, r^2 = r^* = r, k^2 = k^* = k, \|rk\| \leq \sqrt{1-\lambda^{-2}} \right\rangle$$

By [8, Theorem 3.2],

$$\mathcal{A} \quad \cong_{\mathbf{1C^*}} \quad \begin{cases} \mathbb{C}^2, & \lambda = 1, \\ \begin{bmatrix} C[0,1] & C_0(0,1] \\ C_0(0,1] & C[0,1] \end{bmatrix}, & \lambda > 1. \end{cases}$$

Theorem (Tietze Theorem for $\mathbf{1C}^*$, [5])

Given unital C-algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \cong_{\mathbf{1C}^*} \mathcal{B}$ iff there is a sequence of four Tietze transformations changing the presentation of $\mathcal{A}$ into the presentation for $\mathcal{B}$.*

Corollary (Elementary Version, [5])

Given finitely presented unital C-algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \cong_{\mathbf{1C}^*} \mathcal{B}$ iff there is a finite sequence of elementary Tietze transformations changing the presentation of $\mathcal{A}$ into the presentation for $\mathcal{B}$.*

- Analytic/continuous relations
  ($\sin(x) = 0$, etc.)
- Formalize familiar universal constructions.
  (free product, tensor product, etc.)
- Characterization of properties.
  (projectivity, separability, etc.)
- Other categories of interest
  (Banach algebras, operator algebras, etc.)

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