Isomorphisms of non-commutative domain algebras

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October 15, 2011

Isomorphisms of Domain algebras

1. [AL1] Isomorphisms of non-commutative domain algebras, JOT (to appear)
2. [AL2] Isomorphisms of non-commutative domain algebras II (submitted)

- Combines ideas from operator algebras and several complex variables
- Non-commutative Domain Algebras: Multivariate, non commutative operator algebras (Popescu, 2007)
- Several complex variables
  - Cartan's Lemma, 1932
  - Thullen's characterization of bounded Reinhardt domains in $\mathbb{C}^2$ with non-compact automorphism group, 1931
  - Sunada's theorem, 1978

Setting: the Full Fock space

- H is a Hilbert space

$$\mathcal{F}^2(H) = \mathbb{C} \oplus H \oplus H^{\otimes 2} \oplus H^{\otimes 3} \oplus \cdots .$$

- If H is n - dimensional, then $\mathcal{F}^2(H) = \ell_2(\mathbb{F}_n^+)$

- $\mathbb{F}_n^+$ is the free semigroup with n generators: $g_1, g_2, \ldots, g_n$.

- $\ell_2(\mathbb{F}_n^+)$ has orthonormal basis $\{\delta_\alpha : \alpha \in \mathbb{F}_n^+\}$.

- Left Creation Operators: isometries with orthogonal ranges

$$S_1, S_2, \ldots, S_n : \ell_2(\mathbb{F}_n^+) \to \ell_2(\mathbb{F}_n^+), \qquad S_i(\delta_\alpha) = \delta_{g_i\alpha}.$$

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- We use $\mathbb{F}_n^+$ to represent arbitrary products: $X_1, X_2, \ldots, X_n$ are variables and $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k} \in \mathbb{F}_n^+$ then

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$$X_\alpha = X_{i_1} X_{i_2} \ldots X_{i_k}.$$

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Model for row contractions (Popescu, 1991, VN Inequality)

If $T_1, \ldots, T_n \in B(H)$ and $\sum_{i \leq n} T_i T_i^* \leq I$, then there exists a unital completely contractive homomorphism $\Phi : \mathcal{A}_n \rightarrow B(H)$ satisfying

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A simple proof for this uses Poisson Transforms, an explicit dilation.

Completely contractive representations

- The unital completely contractive representations of $\mathcal{A}_n$ on $B(H)$ are given by

$$\left\{ (T_1, \ldots, T_n) : T_i \in B(H) \text{ and } \sum_{i \le n} T_i T_i^* \le I. \right\}$$

- When $\dim(H) = 1$, we obtain the characters

$$B_n = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \|\lambda\|_2 = \sqrt{\sum_{i \le n} |\lambda_i|^2} \le 1 \right\}.$$

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The techniques and methods used to study $\mathcal{A}_n$ are associated with multivariable interpolation results (Caratheodory, Nevanlinna-Pick)

A not pretty generalization: Arias and Popescu, 1999

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Weighted shifts

- For $i \leq n$, define $W_i : \ell_2(\mathbb{F}_n^+) \to \ell_2(\mathbb{F}_n^+)$ by $W_i \delta_\alpha = \sqrt{\frac{\omega_{\beta\alpha}}{\omega_\alpha}} \delta_{g_i \alpha}$, where
- $\omega_\alpha > 0$ for all $\alpha \in \mathbb{F}_n^+$ and $\frac{\omega_{\alpha\beta\gamma}}{\omega_{\beta\gamma}} \leq \frac{\omega_{\alpha\beta}}{\omega_\beta}$ for any $\alpha, \beta, \gamma \in \mathbb{F}_n^+$ with $|\alpha| = |\gamma| = 1$.
- The weights are motivated by a paper of Quiggin (interpolation).
- Then there exist a_α's satisfying

$$\begin{cases} a_{g_i} > 0 \text{ for } i \leq n, \\ a_\alpha \geq 0 \text{ for } |\alpha| \geq 1 \text{ such that} \end{cases}$$

- $\sum_{|\alpha| \geq 1} a_\alpha W_\alpha W_\alpha^* \leq I$, and $(W_1, \ldots, W_n)$ is the model for operators satisfying $\sum_{|\alpha| \geq 1} a_\alpha T_\alpha T_\alpha^* \leq I$
- The Poisson transform works!

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then

- There exist weighted shifts $W_1^f, W_2^f, \ldots, W_n^f$ on the Full Fock space $\ell_2\left(\mathbb{F}_n^+\right)$ that are the model for operator satisfying

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- The symbol $f = \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ is used to identify the operators. $X_1, \ldots, X_n$ are free variables and f is called positive regular n-free formal power series.

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$$\mathcal{D}_f(H) = \left\{ (T_1, \ldots, T_n) : T_i \in B(H) \text{ and } \sum_{|\alpha| \geq 1} a_\alpha T_\alpha T_\alpha^* \leq I. \right\}$$

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- When $\dim(H) = 1$, we obtain the characters, which are given by the bounded domain:

$$\mathcal{D}_f(\mathbb{C}) = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \sum_{|\alpha| \geq 1} a_\alpha |\lambda_\alpha|^2 \leq 1 \right\}.$$

Example

$f = X_1 + X_2$. Then $W_1^f = S_1$ and $W_2^f = S_2$

- $\{S_1, S_2\}$ is the model for row contractions $T_1 T_1^* + T_2 T_2^* \leq I$
- The set of 1-dimensional representations (characters) of $\mathcal{A}(\mathcal{D}_f)$ is

$$\mathcal{D}_f(\mathbb{C}) = \left\{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 : |\lambda_1|^2 + |\lambda_2|^2 \leq 1 \right\}.$$

Example

$g = X_1 + X_2 + X_1X_2$. Then we obtain $\mathcal{A}(\mathcal{D}_g)$

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- $\mathcal{D}_g(\mathbb{C}) = \left\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : |\lambda_1|^2 + |\lambda_2|^2 + |\lambda_1\lambda_2|^2 \leq 1\right\}$.

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$h = X_1 + X_2 + \frac{1}{2}X_1X_2 + \frac{1}{2}X_2X_1$. Then we obtain $\mathcal{A}(\mathcal{D}_h)$

- $\{W_1^h, W_2^h\}$ is the model for operators $T_1T_1^* + T_2T_2^* + \frac{1}{2}T_1T_2T_2^*T_1^* + \frac{1}{2}T_2T_1T_1^*T_2^* \leq I$.
- $\mathcal{D}_h(\mathbb{C}) = \left\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : |\lambda_1|^2 + |\lambda_2|^2 + \frac{1}{2}|\lambda_1\lambda_2|^2 + \frac{1}{2}|\lambda_2\lambda_1|^2 \leq 1\right\}$
- $\mathcal{D}_h(\mathbb{C}) = \left\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : |\lambda_1|^2 + |\lambda_2|^2 + |\lambda_1\lambda_2|^2 \leq 1\right\} = \mathcal{D}_g(\mathbb{C})$

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- For $k \geq 1$, $\mathcal{D}_g\left(\mathbb{C}^k\right) \xrightarrow{\widehat{\Phi}_k} \mathcal{D}_f\left(\mathbb{C}^k\right) \subset \mathbb{C}^{nk^2}$ is biholomorphic.

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- Biholomorphic maps are "rigid"

Theorem (Cartan's Lemma, 1932)

Let D_1, D_2 be bounded circular domains of $\mathbb{C}^k$, $k \geq 2$, containing the origin. If $\psi : D_1 \to D_2$ is biholomorphic and $\psi(0) = 0$, then ψ is the restriction of a linear map.

Theorem (Thullen, 1931)

Let D be a bounded Reinhardt domain of $\mathbb{C}^2$. If there exists $\psi \in Aut(D)$ such that $\psi(0) \neq 0$ (equivalently, $Aut(D)$ is not compact), then D is a ball, a polydisc, or a set of the form $\left\{(z, w) : |z|^2 + |w|^{2/p} \leq 1\right\}$, $p \neq 1$.

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- $\mathcal{D}_g(\mathbb{C}) = \mathcal{D}_h(\mathbb{C})$!

Theorem (Main result of [AL1])

If $\Phi : \mathcal{A}(\mathcal{D}_f) \mapsto \mathcal{A}(\mathcal{D}_g)$ *is a unital completely contractive isomorphism and* $\widehat{\Phi}_1(0) = 0$, *then there exist an invertible matrix* $M = [m_{ij}]$ *such that for* $i \leq n$,

$$\Phi\left(W_i^f\right) = m_{i1}W_1^g + m_{i2}W_2^g + \cdots m_{in}W_n^g.$$

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- The cubic term is zero - uses Cartan's Lemma applied to representation on $\mathbb{C}^3$, etc. etc.

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- $h = X_1 + X_2 + \frac{1}{2} X_1 X_2 + \frac{1}{2} X_2 X_1$
- Recall that $\mathcal{D}_g(\mathbb{C}) = \mathcal{D}_h(\mathbb{C}) \subset \mathbb{C}^2$
- Assume $\Phi : \mathcal{A}(\mathcal{D}_g) \to \mathcal{A}(\mathcal{D}_h)$ is an isomorphism
- Then $\widehat{\Phi} : \mathcal{D}_h(\mathbb{C}) \to \mathcal{D}_g(\mathbb{C})$ is biholomorphic
- By Thullen $\widehat{\Phi}(0) = 0$
- By the Main Theorem, Φ is very simple and after some work we reach a contradiction. $\mathcal{A}(\mathcal{D}_g)$ and $\mathcal{A}(\mathcal{D}_h)$ are not isomorphic!

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- However, the method was ad hoc and we could not generalize it until [AL2]

- $\mathcal{D}_f(\mathbb{C}) = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \sum_{|\alpha| \geq 1} a_\alpha |\lambda_\alpha|^2 \leq 1 \right\}$
- $\mathcal{D}_f(\mathbb{C})$ is a Reinhardt domain. That is, whenever
  - $(\lambda_1, \ldots, \lambda_n) \in \mathcal{D}_f(\mathbb{C})$, and $\theta_1, \ldots, \theta_n \in \mathbb{R}$, then
  - $\left(e^{i\theta_1}\lambda_1, \ldots, e^{i\theta_n}\lambda_n\right) \in \mathcal{D}_f(\mathbb{C})$.

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Theorem (Sunada)

Let D be a bounded Reinhardt domains of $\mathbb{C}^k$, $k \geq 2$, that contains the origin. Then after rescaling and permuting, we can find several indexes such that

- $D \cap (\mathbb{C}^p \times \{0\})$ *is a product of balls*
- $D \cap (\{0\} \times \mathbb{C}^{n-p}) = \{0\} \times D_1$, *and*
- $(z_1, \ldots, z_r, z_{r+1}, \ldots, z_s) \in D$ *iff* $|z_1| < 1, \ldots, |z_r| < 1$ *and*

$$\left(\frac{z_{r+1}}{\prod_{j=1}^r \left(1 - |z_j|^2\right)^{q_{r+1,j}}}, \ldots, \frac{z_s}{\prod_{j=1}^r \left(1 - |z_j|^2\right)^{q_{s,j}}} \right) \in D_1$$

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Theorem ([AL2])

If f and g are aspherical (not balls) then all biholomorphic maps from $\mathcal{D}_f(\mathbb{C})$ to $\mathcal{D}_g(\mathbb{C})$ fix the origin.

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- And this is a convex combination