Symmetries of Cuntz-Pimsner algebras

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Outline

- We recall the concept of Cuntz-Pimsner algebra, crossed product C^*-correspondences, and the Hao-Ng theorem.
- We give applications to group actions on graph C^*-algebras and make the connection with the Doplicher-Roberts algebras of group representations.
- We illustrate with some examples, including group actions on Hermitian vector bundles.
- We discuss similar results for groupoid actions.

Cuntz-Pimsner algebras

- A C^*-correspondence from A to B is a Hilbert B-module $\mathcal{H}$ with a left multiplication given by $\phi : A \to \mathcal{L}_B(\mathcal{H})$.
- It can be thought as a multivalued, partially defined morphism $A \to B$.
- If $A = B$, the Cuntz-Pimsner algebra $\mathcal{O}_A(\mathcal{H})$ is universal for covariant representations $\pi : A \to C, \tau : \mathcal{H} \to C$

$$\tau(a\xi) = \pi(a)\tau(\xi), \quad \pi(\langle \xi, \eta \rangle) = \tau(\xi)^*\tau(\eta)$$

$$\pi(a) = \psi(\phi(a)) \;\text{ for }\; a \in J_{\mathcal{H}} = \phi^{-1}(\mathcal{K}_A(\mathcal{H})) \cap (\ker \phi)^{\perp},$$

  where $\psi : \mathcal{K}_A(\mathcal{H}) \to C, \psi(\theta_{\xi,\eta}) = \tau(\xi)\tau(\eta)^*$.
- Denote by $(\pi_A, \tau_{\mathcal{H}})$ the universal representation of $(A, \mathcal{H})$.

Examples

- If $\alpha \in \mathrm{Aut}(A)$, then $\mathcal{O}_A(\mathcal{H}_\alpha) \cong A \rtimes_\alpha \mathbb{Z}$.
- If $E = (E^0, E^1, r, s)$ is a (topological) graph and $A = C_0(E^0), \mathcal{H}_E = \overline{C_c(E^1)}$, then $\mathcal{O}_A(\mathcal{H}_E) = C^*(E)$.
- In particular we obtain Cuntz algebras $\mathcal{O}_n$ and Cuntz-Krieger algebras $\mathcal{O}_\Lambda$ related to subshifts of finite type.
- If $\mathcal{E} \to X$ is a complex vector bundle, $A = C(X), \mathcal{H} = \Gamma(\mathcal{E})$, then $\mathcal{O}_A(\mathcal{H}) = \mathcal{O}_\mathcal{E}$ is a (locally trivial) continuous field of Cuntz algebras.
- If $\mathcal{E}$ is a line bundle, then $\mathcal{O}_\mathcal{E}$ is commutative with spectrum homeomorphic to the circle bundle of $\mathcal{E}$.
- For $X = S^{2k}$ and $[\mathcal{E}] = n + mt \in K^0(S^{2k}) \cong \mathbb{Z}[t]/(t^2)$ with $n \geq 3$ and $\gcd(n-1, m) = 1$ we have

$$K_0(\mathcal{O}_\mathcal{E}) \cong \mathbb{Z}/(n-1)^2\mathbb{Z} \neq K_0(C(S^{2k}) \otimes \mathcal{O}_n).$$

Group actions and crossed products

- A locally compact group G acts on $(A, \mathcal{H})$ by (α, β) if

$$\langle \beta_g(\xi), \beta_g(\eta) \rangle = \alpha_g(\langle \xi, \eta \rangle),\ \beta_g(\xi a) = \beta_g(\xi)\alpha_g(a),\ \beta_g(a\xi) = \alpha_g(a)\beta_g(\xi).$$

- By the universal property we get $\gamma : G \to \text{Aut } \mathcal{O}_A(\mathcal{H})$.
- For $a \in C_c(G, A), \xi \in C_c(G, \mathcal{H})$ define

$$(a\xi)(s) = \int_G a(t)\beta_t(\xi(t^{-1}s))dt,\ (\xi a)(s) = \int_G \xi(t)\alpha_t(a(t^{-1}s))dt,$$

$$\langle \xi, \eta \rangle(s) = \int_G \alpha_{t^{-1}}(\langle \xi(t), \eta(ts) \rangle)dt.$$

- The completion gives $(A \rtimes_\alpha G, \mathcal{H} \rtimes_\beta G)$.

Main results

- **Theorem** (Hao-Ng). If G is amenable, then
$$\mathcal{O}_A(\mathcal{H}) \rtimes_\gamma G \cong \mathcal{O}_{A\rtimes_\alpha G}(\mathcal{H} \rtimes_\beta G).$$
If is G abelian, this is $\hat{G}$-equivariant.
- **Lemma 1**. If G acts on $(A, \mathcal{H})$ and $A \rtimes G$ decomposes as $\bigoplus A_i$ with units q_i, then $\mathcal{H} \rtimes G$ decomposes into $\bigoplus q_i(\mathcal{H} \rtimes G)q_j$, a sum of C^*-correspondences from A_i to A_j.
- **Lemma 2**. If $\mathcal{N}$ is an imprimitivity module from A to B and $\mathcal{H}$ is a C^*-correspondence over A, then $\mathcal{H}' = \mathcal{N}^* \otimes_A \mathcal{H} \otimes_A \mathcal{N}$ is a C^*-correspondence over B and $\mathcal{O}_A(\mathcal{H})$ is Morita-Rieffel equivalent to $\mathcal{O}_B(\mathcal{H}')$.
- **Theorem** (D). If G compact acts on a discrete graph $E = (E^0, E^1, r, s)$, then $C^*(E) \rtimes G$ is Morita-Rieffel equivalent to a graph algebra.

Proof

- We have

$$C_0(E^0)\rtimes G \cong \bigoplus C(Gx)\rtimes G \cong \bigoplus C(G/G_x)\rtimes G \cong \bigoplus M_{|Gx|}\otimes C^*(G_x),$$

  which is Morita-Rieffel equivalent to $C_0(V)$ via $\mathcal{N}$, with $V = \{v_i\}$ at most countable.
- Then $\mathcal{M} = \mathcal{N}^* \otimes (\mathcal{H}_E \rtimes G) \otimes \mathcal{N}$ over $C_0(V)$ determines a graph F with $F^0 = V$ and edges F^1 determined by the incidence matrix

$$a_{ij} = \dim p_i \mathcal{M} p_j$$

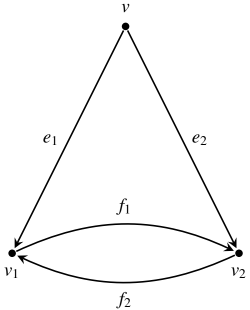
  where $p_i = \chi_{\{v_i\}}$.
- Then

$$C^*(E) \rtimes G \cong \mathcal{O}_{C_0(E^0)\rtimes G}(\mathcal{H}_E \rtimes G)$$

  is Morita-Rieffel equivalent to $C^*(F)$.

Examples

- Let E be



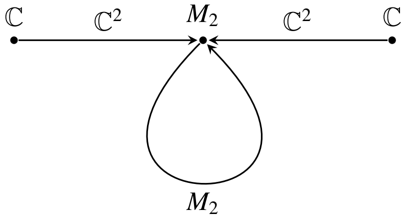
- Here $A = C(E^0) = \mathbb{C}^3, \mathcal{H} = C(E^1) = \mathbb{C}^4$ and $G = \mathbb{Z}_2$ acts by

$$\alpha(a_1, a_2, a_3) = (a_1, a_3, a_2),\ \beta(\xi_1, \xi_2, \xi_3, \xi_4) = (\xi_2, \xi_1, \xi_4, \xi_3).$$

Examples

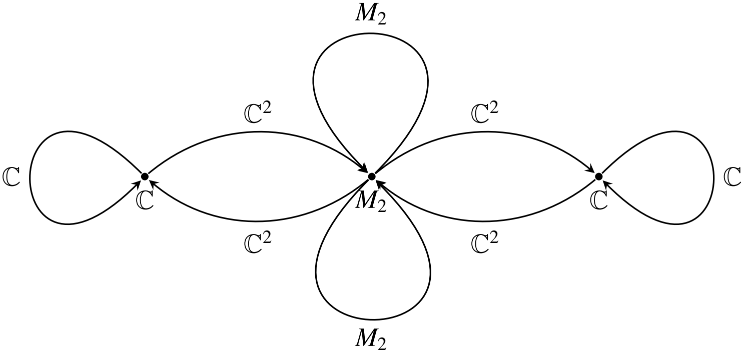
- Then $A \rtimes_\alpha \mathbb{Z}_2 \subset M_2(A)$ as $\begin{bmatrix} a & b \\ \alpha(b) & \alpha(a) \end{bmatrix}$ and $\mathcal{H} \rtimes_\beta \mathbb{Z}_2 \subset M_2(\mathcal{H})$ as $\begin{bmatrix} \xi & \eta \\ \beta(\eta) & \beta(\xi) \end{bmatrix}$ with obvious operations.

- Since $A \rtimes \mathbb{Z}_2 \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2$, the crossed product $\mathcal{H} \rtimes \mathbb{Z}_2$ decomposes as $\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus M_2$ and $C^*(E) \rtimes \mathbb{Z}_2$ is given by



Examples

- Let E be the graph with one vertex and three loops which gives $\mathcal{O}_3$.
- The symmetric group S_3 acts trivially on $A = \mathbb{C}$ and by the permutation representation ρ on $\mathcal{H} = \mathbb{C}^3$.
- $\mathcal{H} \rtimes S_3$ decomposes over $A \rtimes S_3 \cong C^*(S_3)$ and $\mathcal{O}_3 \rtimes S_3$ is given by



- Recall that $\hat{S_3} = \{\iota, \varepsilon, \sigma\}, \ \rho \sim \iota + \sigma$, and the Doplicher-Roberts algebra $\mathcal{O}_\rho$ constructed from intertwiners (ρ^n, ρ^m) is Morita-Rieffel equivalent to the same graph algebra, using the character table of S_3.

Examples

- Consider a hermitian G-vector bundle $\mathcal{E} \to X$. If X is a point, then $\mathcal{E}$ is a G-module. If X has trivial action, then $\mathcal{E}$ is a family of G-modules.
- The group G acts on $\Gamma(\mathcal{E})$ by

$$(g\xi)(x) = g\xi(g^{-1}x)$$

  and we may study $\mathcal{O}_{\mathcal{E}} \rtimes G \cong \mathcal{O}_{C(X)\rtimes G}(\Gamma(\mathcal{E}) \rtimes G)$
- **Fact 1**. If G compact acts freely on $\mathcal{E} \to X$, then $\mathcal{O}_{\mathcal{E}} \rtimes G$ is a continuous field of Cuntz algebras over X/G.
- **Fact 2**. If G acts fiberwise on $\mathcal{E} \to X$ of rank n, then $\mathcal{O}_{\mathcal{E}} \rtimes G$ is a continuous field with fibers $\mathcal{O}_n \rtimes G$.
- If X is a manifold and G acts by diffeomorphisms, then $\mathcal{E} = TX \otimes \mathbb{C}$ becomes a G-bundle. What is $\mathcal{O}_{\mathcal{E}} \rtimes G$?

Groupoid actions

- Recall that a $C_0(X)-$algebra is a C^*-algebra A together with a homomorphism $\theta : C_0(X) \to ZM(A)$ such that $\overline{\theta(C_0(X))A} = A$.
- For each $x \in X$ we define the fiber A_x as $A/\overline{\theta(I_x)A}$ where

$$I_x = \{f \in C_0(X) : f(x) = 0\}.$$

- A $C_0(X)$-algebra A gives rise to an upper semicontinuous C^*-bundle $\mathcal{A}$ such that $A \cong \Gamma_0(X, \mathcal{A})$.
- We say that a groupoid G acts on a $C_0(G^0)$-algebra A if for each $g \in G$ there is an isomorphism $\alpha_g : A_{s(g)} \to A_{r(g)}$ such that

$$\alpha_{g_1 g_2} = \alpha_{g_1} \circ \alpha_{g_2} \text{ for } (g_1, g_2) \in G^{(2)}.$$

- If the groupoid G acts on X, then $C_0(X)$ becomes in a natural way a $C_0(G^0)$-algebra and G acts on $C_0(X)$ by $\alpha_g(f)(x) = f(g^{-1} \cdot x)$.
- Groupoid actions on elementary C^*-bundles over G^0 satisfying Fell's condition appear in the context of defining the Brauer group $\mathrm{Br}(G)$.

$C_0(X)$- C^*-correspondences

- Let $\mathcal{H}$ be a Hilbert module over a $C_0(X)$-algebra A. Define the fibers $\mathcal{H}_x := \mathcal{H} \otimes_A A_x$ which are Hilbert A_x-modules.
- The set $\text{Iso}(\mathcal{H})$ of $\mathbb{C}$-linear isomorphisms between fibers becomes a groupoid with unit space X.
- We say that $\mathcal{H}$ is a $C_0(X)$-C^*-correspondence over A if there is a $*$-homomorphism $\phi : A \to \mathcal{L}_A(\mathcal{H})$ such that

$$\phi(fb)(\xi a) = \phi(b)(\xi fa) \text{ for all } f \in C_0(X),\ a, b \in A,\ \xi \in \mathcal{H}.$$

- Each fiber $\mathcal{H}_x$ becomes a C^*-correspondence over A_x.
- An action of G on $\mathcal{H}$ is given by a homomorphism $\rho : G \to \text{Iso}(\mathcal{H})$ where $\rho_g : \mathcal{H}_{s(g)} \to \mathcal{H}_{r(g)}$ with $\rho_g = I$ if $g \in G^0$, such that

$$\langle \rho_g \xi, \rho_g \eta \rangle_{r(g)} = \alpha_g(\langle \xi, \eta \rangle_{s(g)}),$$

  for $\xi, \eta \in \mathcal{H}_{s(g)}$ and

$$\rho_g(\xi a) = \rho_g(\xi)\alpha_g(a), \quad \rho_g(a\xi) = \alpha_g(a)\rho_g(\xi)$$

  for $\xi \in \mathcal{H}_{s(g)},\ a \in A_{s(g)}$.

Crossed products

- **Proposition**. Suppose the groupoid G with Haar system $\{\lambda^x\}$ acts on the $C_0(G^0)$-C^*-correspondence $\mathcal{H}$ over A.
- Then the completion of $\Gamma_c(G, r^*\mathcal{H})$ becomes a C^*-correspondence $\mathcal{H} \rtimes G$ over $A \rtimes G$, using the inner product

$$\langle \xi, \eta \rangle(g) = \int_G \alpha_h(\langle \xi(h^{-1}), \eta(h^{-1}g)\rangle_{s(h)})d\lambda^{r(g)}(h),$$

and the multiplications

$$(\xi f)(g) = \int_G \xi(h)\alpha_h(f(h^{-1}g))d\lambda^{r(g)}(h),$$

$$(f\xi)(g) = \int_G f(h)\rho_h(\xi(h^{-1}g))d\lambda^{r(g)}(h),$$

where $\xi, \eta \in \Gamma_c(G, r^*\mathcal{H}), f \in \Gamma_c(G, r^*A)$.

Hao-Ng for groupoids

- **Theorem** (D). Suppose G acts on a $C_0(G^0)$-C^*-correspondence $\mathcal{H}$ over A. Then the Katsura ideal $J_{\mathcal{H}}$ is G-invariant and G acts on $\mathcal{O}_A(\mathcal{H})$, which becomes a $C_0(G^0)$-algebra with fibers $\mathcal{O}_{A_x}(\mathcal{H}_x)$ for $x \in G^0$.
- Let G amenable act on a $C_0(G^0)$-C^*-correspondence $\mathcal{H}$ over A. Assume that $J_{\mathcal{H}\rtimes G} \cong J_{\mathcal{H}} \rtimes G$. Then there are maps

$$\Phi : A \rtimes G \to \mathcal{O}_A(\mathcal{H}) \rtimes G,\ \Phi(f)(g) = \pi_A(f(g)),$$

$$\Psi : \mathcal{H} \rtimes G \to \mathcal{O}_A(\mathcal{H}) \rtimes G,\ \Psi(\xi)(g) = \tau_{\mathcal{H}}(\xi(g))$$

  which induce an isomorphism

$$\mathcal{O}_{A\rtimes G}(\mathcal{H} \rtimes G) \cong \mathcal{O}_A(\mathcal{H}) \rtimes G.$$

- **Corollary**. A vector bundle $p : \mathcal{E} \to X$ is called a G-bundle if $G^0 = X$ and there is a groupoid morphism $G \to \mathrm{Iso}(\mathcal{E})$.
- $\mathcal{O}_{\mathcal{E}} \rtimes G$ is the Cuntz-Pimsner algebra of the C^*-correspondence $\Gamma(\mathcal{E}) \rtimes G$ over $C^*(G)$.

Selected references

- V. Deaconu, *Group actions on graphs and C^*-correspondences*, Houston J. Math vol. 44 No. 1 (2018), 147–168.
- V. Deaconu, A. Kumjian, D. Pask and A. Sims, *Graphs of C^*-correspondences and Fell bundles,* Indiana Univ. Math. J. 59 (2010) 1687–1735.
- G. Hao, C.-K. Ng, *Crossed products of C*-correspondences by amenable group actions*, J. Math. Anal. Appl. 345 (2008), no. 2, 702–707.
- A. Kumjian, D. Pask, I. Raeburn, J. Renault, *Graphs, Groupoids and Cuntz-Krieger algebras*, J. of Funct. Anal. 144 (1997) No. 2, 505–541.
- M.H.Mann, I. Raeburn, C.E. Sutherland, *Representations of finite groups and Cuntz-Krieger algebras*, Bull. Austral. Math. Soc. 46(1992), 225–243.
- G. Segal, *Equivariant K-theory*, Inst. Hautes Études Sci. Publ. Math. No. 34 1968, 129–151.
- E. Vasselli, *The C^*-algebra of a vector bundle and fields of Cuntz algebras*, J. Funct. Anal. 222 (2005), no. 2, 491–502.