Perfect bandlimited reconstruction

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Denote by $\mathcal{B}_p(\tau)$ $(1 \leq p \leq 2)$ the class of entire functions $F \in L^p(\mathbb{R})$ so that

$$\widehat{F}(t) = 0, \qquad |t| \geq \tau.$$

Goal. Construction of bandlimited φ so that

$$\int_T |G(x)|^p dx \leq \|\varphi\|_2^2 \, \rho(T, \delta) \int_{\mathbb{R}} |G(x)|^p dx$$

with the Nyquist density

$$\rho(T, \delta) = \sup_{x \in \mathbb{R}} |T \cap [x, x + \delta]|.$$

(Notation: $P_T G = \chi_T G$.)

Outline

1. Motivation: Reconstruction of corrupted signals
2. Construction of φ
3. Multivariate generalizations

Reconstruction of noisy signals ($p = 1$)

Problem.

- F signal,
- n corrupting noise,
- Known: $F + n$.

Is it possible to recover F ?

Modelling.

- F is an element of $\mathcal{B}_1(\tau)$
- $n \in L^1(\mathbb{R})$ is 'sparse'

Ill-posed problem without the (unspecified) sparsity condition.

Approximations

- Use the best L^1-approximation of $F + n$ from $\mathcal{B}_1(\tau)$ as recovery candidate.
- Notation: For $f \in L^1(\mathbb{R})$ define $f^* \in \mathcal{B}_1(\tau)$ by

$$\|f - f^*\|_1 \leq \|f + G\|_1$$

for all $G \in \mathcal{B}_1(\tau)$, if such f^* exists. (Abuse of notation; f^* not always unique.)

Sparsity. Frequently given as condition on $T = \text{supp}(n)$ such that

$$(F + n)^* = F.$$

- Equivalent: Condition for $n^* \equiv 0$.

Badly approximable support

Lemma. (Logan 1965) Let $T \subseteq \mathbb{R}$. If

$$\int_T |G| < \frac{1}{2} \|G\|_1$$

for all $G \in \mathcal{B}_1(\tau)$, then $n^* \equiv 0$ for functions with $\mathrm{support}(n) \subseteq T$.

- Independently rediscovered by Benyamini, Kroo, Pinkus (2012)
- Related to compressed sensing

Proof

Let $G \in \mathcal{B}_1(\tau)$, not the zero function. By assumption $\int_T |G| < \frac{1}{2} \int_{\mathbb{R}} |G|$, i.e.,

$$\int_T |G| < \int_{T^c} |G|.$$

Let n have support T. Then

$$\begin{aligned} \|n - G\|_1 &= \int_T |n - G| + \int_{T^c} |G| \\ &\geq \int_T |n| - \int_T |G| + \int_{T^c} |G| \\ &> \int_T |n| \end{aligned}$$

and this is $\|n\|_1$. Hence none of the non-zero functions from $\mathcal{B}_1(\tau)$ gives a better approximation than the zero function.

In other words: If for some $\delta > 0$ and all $G \in \mathcal{B}_1(\tau)$

$$\int_T |G(x)| dx \leq C\,\rho(T,\delta) \int_{\mathbb{R}} |G(x)| dx,$$

then perfect reconstruction of F from $F + n$ is possible if

$$\rho(T,\delta) < \frac{1}{2C}.$$

(Recall: $\rho(T,\delta) = \sup_{u \in \mathbb{R}} |T \cap [u, u+\delta]|$.)

Reconstruction of missing data, $p = 2$

Given $G \in \mathcal{B}_2(\tau)$ with the property that the values of G on $T \subseteq \mathbb{R}$ are missing (where T is known). Is it possible to recover G?

Donoho, Stark (1985):

$$\|P_T\|_{\mathcal{B}_2(\tau)} < 1$$

implies that recovery is possible.

Missing data of $G \in \mathcal{B}_2(\tau)$: Model as $r = (I - P_T)G$, and note that

$$(I - P_T)G = (I - P_T L_\tau)G.$$

where $L_\tau = \mathcal{F}^{-1} P_{[-\tau,\tau]} \mathcal{F}$ with the Fourier transform $\mathcal{F}$. Then formally

$$G - (I - P_T L_\tau)^{-1} r = G - G = 0,$$

so reconstruction is (in principle) possible if

$$\|P_T L_\tau\|_{L^2 \to L^2} < 1.$$

(They developed an algorithm as well.) They observed

$$\|P_T L_\tau\|_{L^2 \to L^2} \leq \|P_T\|_{\mathcal{B}_2(\tau) \to L^2},$$

hence require the operator norm on the right to be < 1.

Bounds for operator norms

Lemma. (Donoho, Logan)

$$\int_T |G|^2 \leq \|\varphi\|_2^2 \, \rho(T, 2\delta) \, \|G\|_2^2,$$

where φ satisfies

- support of φ is contained in $[-\delta, \delta]$,
- $T_\varphi(G) = G * \varphi$ is a continuously invertible transformation on $\mathcal{B}_2(\tau)$ with

$$\|T_\varphi^{-1}\| = 1.$$

Modification for $p = 1$:

$$\int_T |G| \leq \|\varphi\|_\infty \rho(T, 2\delta) \|G\|_1$$

where φ satisfies

- support of φ is contained in $[-\delta, \delta]$,
- $T_\varphi(G) = G * \varphi$ is a continuously invertible transformation on $\mathcal{B}_1(\tau)$ with
$$\|T_\varphi^{-1}\| = 1.$$

$p = 2$

With F defined by $F = T_\varphi^{-1}(G)$ we have with $\chi_B = \chi_{[-\delta,\delta]}$

$$\begin{aligned}
\int_T |G|^2 &= \int_T \left| \int_{\mathbb{R}} \varphi(u) F(x-u) du \right|^2 dx \\
&\leq \|\varphi\|_2^2 \int_T \int_{\mathbb{R}} \chi_B(u) |F(x-u)|^2 du dx \\
&= \|\varphi\|_2^2 \int_{\mathbb{R}} |F(u)|^2 \int_{[u-\delta, u+\delta]} \chi_T(x) dx du \\
&\leq \|\varphi\|_2^2 \sup_{u \in \mathbb{R}} |[u, u+2\delta] \cap T| \, \|G\|_2^2.
\end{aligned}$$

Question. How small can we make $\|\varphi\|_2$?

Conditions on φ

1. Support of φ is $\subseteq [-\delta, \delta]$,
2. The operator $T_\varphi : B_2(\tau) \to B_2(\tau)$ defined by $G \mapsto \varphi * G$ has continuous inverse,
3. $\|T_\varphi^{-1}\| = 1$.

Goal. Minimize $\|\varphi\|_2$.

- Model $F = \widehat{\varphi}$, or
- Model $H = FF^\#$.

Lemma. (Akhiezer) If H of type 2δ is nonnegative on $\mathbb{R}$, then $H = FF^\#$ where F has type δ.

(Donoho and Logan: $|\tau| \leq 1/2$ and $2\tau \in \mathbb{N}$, Littmann: arbitrary $\tau > 0$)

$|\tau| \leq 1/2$; normalize $\delta = \pi$

Sufficient condition. Model $\|F\|_2$. Start with

$$F(x) = \int_{-1}^{1} \widehat{F}(t) \cos(\pi x t) dt.$$

Cauchy-Schwarz: Set $C(t) = \chi_{[-1,1]}(t) \cos \pi \tau t$.

$$1 = |F(\tau)|^2 \leq \|\widehat{F}\|_2^2 \|C\|_2^2,$$

with equality if $\widehat{F}(t) = \cos \pi \tau t$ on $[-1, 1]$.

F turns out to have the correct properties for $\tau \leq 1/2$; for $\tau > 1/2$ this will not produce the optimal function.

Arbitrary τ, $\delta = \pi$

Needed:

1. H has type 2π,
2. $H(\pm\tau) = 1$,
3. Sufficient condition should represent $\int_{\mathbb{R}} H(x)dx$ as a sum of values over a discrete set that includes $\pm\tau$.

Hardy Space

The Hardy space $H^2(\mathbb{C}^+)$ is the vector space of functions F analytic in the upper half plane with

$$\sup_{y>0} \|F_y\|_2 < \infty$$

where $F_y(x) = F(x + iy)$.

Fact. Functions $F \in H^2(\mathbb{C}^+)$ are in isometric correspondence to functions $f \in L^2(\mathbb{R})$ with

$$\widehat{f}(t) = 0 \qquad \text{if } t < 0.$$

De Branges spaces

An entire function E is called Hermite-Biehler, if for $\Im z > 0$,

$$|E(z)| > |E^{\#}(z)|$$

where $E^{\#}(z) = \overline{E(\overline{z})}$. We set

$$\mathcal{H}(E) = \{F \text{ entire } : F/E, F^{\#}/E \in H^2(\mathbb{C}^+)\}.$$

We define real entire A, B by $E(z) = A(z) - iB(z)$.

Fact. $\mathcal{H}(E)$ is a reproducing kernel Hilbert space with kernel

$$K(w,z) = \frac{B(z)A(\bar{w}) - A(z)B(\bar{w})}{\pi(z - \bar{w})}.$$

De Branges Theorem

Let $\mathcal{T}_B$ be the set of (real) zeros of B. Then for $F \in \mathcal{H}(E)$

$$\int_{\mathbb{R}} |F(x)|^2 \frac{dx}{|E(x)|^2} = \sum_{t \in \mathcal{T}_B} \frac{|F(t)|^2}{K(t,t)}.$$

If $H = FF^{\#}$, then this becomes

$$\int_{\mathbb{R}} H(x) \frac{dx}{|E(x)|^2} = \sum_{t \in \mathcal{T}_B} \frac{H(t)}{K(t,t)}$$

Consider $E(z) = e^{-i\pi z}(z + i\gamma)$ where $\gamma > 0$. Then E is Hermite-Biehler, and $E = A - iB$ where

$$B(z) = z \sin \pi z - \gamma \cos \pi z.$$

Happy circumstance: if $\gamma = \tau \sin \pi\tau / \cos \pi\tau$, then

$$B(\pm\tau) = 0.$$

(Need to modify E if γ is not positive.)

Drawback: $dx/|E(x)|^2$ is not Lebesgue measure.

We have

$$\frac{dx}{|E(x)|^2} = \frac{dx}{x^2 + \gamma^2}$$

Define $S(z) = H(z)(z^2 + \gamma^2)$, and note

$$\int_{\mathbb{R}} H(x)dx = \int_{\mathbb{R}} S(x)\frac{dx}{|E(x)|^2}.$$

Use the de Branges theorem on the right side to get the required quadrature formula:

Quadrature

For every $G \in \mathcal{B}_1(2\pi)$,

$$\int_{\mathbb{R}} G(x)dx = \sum_{\xi \in \mathcal{T}_\gamma} \left(1 - \frac{\gamma}{\pi(\xi^2 + \gamma^2) + \gamma}\right) G(\xi)$$

where $\mathcal{T}_\gamma$ is the zero set of

$$B(z) = z \sin \pi z - \gamma \cos \pi z$$

or

$$A(z) = z \cos z + \gamma \sin z.$$

Lemma. $H = FF^{\#}$ is optimal if

$$H(\xi) = \chi_{[-\tau,\tau]}(\xi)$$

for all $\xi \in \mathcal{T}_B$, and $H \geq \chi_{[-\tau,\tau]}$ on $\mathbb{R}$.

Interpolation

- Can define an interpolation I so that

$$I(z) - \chi_{[-\tau,\tau]}(z) = B^2(z)R(z),$$

  where the remainder R is a nonpositive function.
- R is a one-sided Laplace transform of g, where

$$\frac{1}{B^2(z)} = \int_{\mathbb{R}} e^{-zt} g(t)dt.$$

Missing case: $2\tau \in \mathbb{N}$.

- Use Poisson summation instead of the de Branges theorem (Selberg, 1960's)
- Work in the classical Paley Wiener space $PW_{2\pi}$

Bounds:

- $2\tau \in \mathbb{N}$: $\|\varphi\|_2^2 = 2\tau + \delta^{-1}$
- $|\tau| \leq 1/2$: $4\pi(2\pi\delta + \frac{\sin 2\pi\delta\tau}{\tau})^{-1}$
- arbitrary τ: more complicated expression $< 2\tau + \delta^{-1}$.

Auxiliary function for $p = 1$

Donoho, Logan: Define

$$M = \{\nu \in \mathcal{M}(\mathbb{R}) : \widehat{\nu}(t) = 1/\widehat{\varphi}(t)\}$$

for $|t| \leq \tau$. Then every $\nu \in M$ is a convolution inverse of φ on $\mathcal{B}_1(\tau)$. If there exists $t_0 \in [-\tau, \tau]$ and $\nu_0 \in M$ with

$$1/|\widehat{\varphi}(t_0)| = \|\nu_0\|,$$

then $\|\nu_0\|$ (total variation) is the operator norm of T_φ^{-1}.

- They calculate the extremal element from M if $\varphi = \chi_{[-\delta,\delta]}$
- Need to minimize $\|\varphi\|_\infty$, this has as a consequence that the characteristic function is optimal for small values of τ.

Multivariate versions (ongoing work with S. Husein)

- $H \subseteq \mathbb{R}^d$ symmetric convex body (centrally symmetric, compact, convex set with nonempty interior),
- $\mathcal{B}_p(H)$ entire functions in $L^p(\mathbb{R}^d)$ with $\widehat{F}(t) = 0$ for $t \notin H$,
- Support of φ is a symmetric convex body W
- Trivial case: H and W cubes,
- Optimal answer can be computed if H and W are discs; leads to one-dimensional problem with respect to measures $|x|^{n-1}dx$