Compression, Matrix Range and Completely Positive Map

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Iowa-Nebraska Functional Analysis Seminar
November 5, 2016

Definitions and notations

$\mathcal{H},\ \mathcal{K}$: Hilbert space. If dim $\mathcal{H} = n < \infty,\ \mathcal{H} \cong \mathbb{C}^n$.

$\mathcal{B}(\mathcal{H}, \mathcal{K})$: bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. If dim $\mathcal{H} = m$ and dim $\mathcal{K} = n$, $\mathcal{B}(\mathcal{H}, \mathcal{K}) \cong M_{n,\,m}$. $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$ and $M_n = M_{n,\,n}$.

$A \in \mathcal{B}(\mathcal{H})$ is said to be self-adjoint if $A = A^*$. $\mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ will denote the space of self-adjoint operators in $\mathcal{B}(\mathcal{H})$.

Every $A \in \mathcal{B}(\mathcal{H})$ has a self-adjoint decomposition $A = A_1 + iA_2$, $A_1,\ A_2 \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$.

If dim $\mathcal{H} = n$, $\mathcal{B}(\mathcal{H})_{\mathrm{sa}} = H_n$, the set of $n \times n$ Hermitian matrices.

$S \subseteq \mathbb{R}^n,\ \mathbb{C}^n$ is said to be **convex** if for all $\mathbf{x},\ \mathbf{y} \in S$, the line segment $\{t\mathbf{x} + (1-t)\mathbf{y} : 0 \leq t \leq 1\} \subseteq S$.

Suppose $A \in \mathcal{B}(\mathcal{H})$ and $\mathcal{K}$ is a norm closed subspace of $\mathcal{H}$. Let $P_{\mathcal{K}} \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$. Then $B = P_{\mathcal{K}} A|_{\mathcal{K}} \in \mathcal{B}(\mathcal{K})$ is called a **compression** of A to $\mathcal{K}$ and A is a **dilation** of B to $\mathcal{H}$.

Let $A \in M_n$ and $1 \leq m \leq n$. Then $B \in M_m$ is a compression of A if and only if there exists $V \in M_{n,m}$ such that $V^*V = I_m$ and $B = V^*AV$.

For $A \in \mathcal{B}(\mathcal{H})$ and $m \geq 1$, let

$$W_m(A) = \{B \in M_m : B \text{ is a compression of } A\}$$

For $m = 1$, $W_1(A) = \{\langle Ax, x\rangle : x \in \mathcal{H},\ \langle x, x\rangle = 1\}$ is the **numerical range** of A, usually denoted by $W(A)$.

By the Toeplitz- Hausdorff Theorem, $W(A)$ is convex. For $m > 1$, $W_m(A)$ is usually not convex.

Compression of linear operators

Theorem 1 (Sz.-Nagy and Foias)

$A \in \mathcal{B}(\mathcal{K})$ is a contraction ($\|A\| \le 1$) if and only if A has a unitary dilation $U \in \mathcal{B}(\mathcal{H})$ such that

$$A^k = P_{\mathcal{K}} U^k \big|_{\mathcal{K}} \text{ for all } k \ge 1.$$

Given $A \in \mathcal{B}(\mathcal{H})$, the **numerical radius** of A is given by

$$w(A) = \sup\{|z| : z \in W(A)\}.$$

$w(A)$ is a norm on $\mathcal{B}(\mathcal{H})$ and satisfies $w(A) \le \|A\| \le 2w(A)$.

Theorem 2 (Sz.-Nagy and Foias)

$A \in \mathcal{B}(\mathcal{K})$ satisfies $w(A) \le 1$ if and only if there is a unitary $U \in \mathcal{B}(\mathcal{H})$ such that

$$A^k = \mathbf{2}\, P_{\mathcal{K}} U^k \big|_{\mathcal{K}} \text{ for all } k \ge 1.$$

Compression of linear operators

Theorem 3 (Ando, Arveson)

$A \in \mathcal{B}(\mathcal{K})$ satisfies $w(A) \leq 1$ if and only if A is a compression of $\begin{pmatrix} 0 & 2I_{\mathcal{H}} \\ 0 & 0 \end{pmatrix}$ for some $\mathcal{H}$.

Note: $W\left(\begin{pmatrix} 0 & 2I_{\mathcal{H}} \\ 0 & 0 \end{pmatrix}\right) = W\left(\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right) = \{z \in \mathbb{C} : |z| \leq 1\}$

$$w(A) \leq 1 \Leftrightarrow W(A) \subseteq W\left(\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right)$$

Let $A = A_1 + iA_2$ be the self-adjoint decomposition of $A \in \mathcal{B}(\mathcal{K})$. Then

$$W(A) \cong W(A_1, A_2) = \{(\langle A_1 x, x\rangle,\ \langle A_2 x, x\rangle) : x \in \mathcal{K},\ \langle x, x\rangle = 1\} \subset \mathbb{R}^2$$

Given $A_1, \cdots, A_p \in \mathcal{B}(\mathcal{K})_{\mathrm{sa}}$, define the joint numerical range
$W(A_1, \cdots, A_p) = \{(\langle A_1 x, x\rangle,\ , \cdots,\ \langle A_p x, x\rangle) : x \in \mathcal{K},\ \langle x, x\rangle = 1\} \subset \mathbb{R}^p$

Completely positive map

$A \in \mathcal{B}(\mathcal{H})$ is said to be positive ($A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$.

An operator system $\mathcal{S}$ of a C^*-algebra $\mathcal{A}$, is a norm-closed self-adjoint ($\mathcal{S} = \mathcal{S}^*$) subspace S of $\mathcal{A}$ containing $1_{\mathcal{A}}$.

A linear map $\Phi : \mathcal{S} \to \mathcal{B}$ is positive on S if $A \geq 0 \Rightarrow \Phi(A) \geq 0$

$\Phi_k : M_k(\mathcal{S}) \to M_k(\mathcal{B})$, $\Phi_k((A_{ij})) = (\Phi(A_{ij}))$

Φ is k-positive if Φ_k is positive.

Φ is completely positive if Φ is k-positive for all $k \geq 1$.

Theorem 4 (Arveson's Extension Theorem)

Let $\mathcal{A}$ be a unital C^*-algebra and S be an operator system of $\mathcal{A}$. Then every completely positive map from S to a C^*-algebra $\mathcal{B}$ can be extended to a completely positive map from $\mathcal{A}$ to $\mathcal{B}$.

Numerical range and positivity

Suppose $W(A_1, A_2) \subseteq W(B_1, B_2)$.

If $c_0 I + c_1 B_1 + c_2 B_2 \geq 0$, then we have

$$\langle (c_0 I + c_1 B_1 + c_2 B_2) x, x \rangle \geq 0 \text{ for all } x \in \mathcal{H} \text{ with } \langle x, x \rangle = 1$$

$$\Rightarrow \quad c_0 + (c_1, c_2) \cdot (b_1, b_2) \geq 0 \text{ for all } (b_1, b_2) \in W(B_1, B_2)$$

$$\Rightarrow \quad c_0 + (c_1, c_2) \cdot (a_1, a_2) \geq 0 \text{ for all } (a_1, a_2) \in W(A_1, A_2)$$

$$\Rightarrow \quad \langle (c_0 I + c_1 A_1 + c_2 A_2) x, x \rangle \geq 0 \text{ for all } x \in \mathcal{K} \text{ with } \langle x, x \rangle = 1$$

Therefore, the map

$$\Phi(c_0 I + c_1 B_1 + c_2 B_2) = (c_0 I + c_1 A_1 + c_2 A_2) \qquad (1)$$

is positive.

Remark: If $A = A_1 + iA_2$ and $B = B_1 + iB_2$. Then (1) is equivalent to

$$\Phi(c_0 I + c_1 B + c_2 B^*) = (c_0 I + c_1 A + c_2 A^*).$$

Theorem 5 (Stinespring's dilation theorem)

Let $\mathcal{A}$ be a unital C^*-algebra, and let $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a linear map. Then Φ is completely positive if and only if there exist a Hilbert space $\mathcal{K}$, a unital C^*-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, and a bounded operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that

$$\Phi(T) = V^*\pi(T)V \quad \text{for all } T \in \mathcal{A}.$$

Note: Φ is unital if and only if $V^*V = I_{\mathcal{H}}$.

Therefore, A is a compression of $B \otimes I_{\mathcal{H}}$ for some $\mathcal{H}$ if and only if $A = \Phi(B)$ for some unital completely positive map Φ.

Reformulation of Theorem 3

Let $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = B_1 + iB_2,\ B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Theorem 3a

Suppose $A_1,\ A_2 \in \mathcal{B}(\mathcal{H})_{\rm sa}$. Then $W(A_1,\ A_2) \subseteq W(B_1, B_2)$ if and only if the map

$$\Phi(c_0 I_2 + c_1 B_1 + c_2 B_2) = c_0 I_{\mathcal{H}} + c_1 A_1 + c_2 A_2$$

is completely positive.

Theorem 3b

Suppose $A \in \mathcal{B}(\mathcal{H})$. Then the map

$$\Phi(c_0 I_2 + c_1 B + c_2 B^*) = c_0 I_{\mathcal{H}} + c_1 A + c_2 A^*$$

is positive on $\mathrm{span}(I_2, B, B^*)$ if and only if it is completely positive.

Another Proof of Theorem 3b

Theorem 6 (Choi)

Let $\mathcal{S}_2$ be the space of 2×2 complex symmetric matrices. Then every positive map $\Phi : \mathcal{S}_2 \to \mathcal{B}(\mathcal{H})$ is completely positive.

Choi proves the above theorem for finite dimensional $\mathcal{H}$. The infinite dimensional case can be proven from the finite dimensional case.

Theorem 3c

Let $B \in M_2$ and $A \in \mathcal{B}(\mathcal{H})$. Then the map

$$\Phi(c_0 I_2 + c_1 B + c_2 B^*) = c_0 I_{\mathcal{K}} + c_1 A + c_2 A^*$$

is positive on $\mathrm{span}(I_2, B, B^*)$ if and only if it is completely positive.

Proof. Every $B \in M_2$ is unitarily similar to a symmetric matrix S. Let $\mathcal{S} = \mathrm{span}(I_2, S, \ S^*)$. If $\mathcal{S} = \mathcal{S}_2$, the result follows from Theorem 6. If $\mathcal{S} \neq \mathcal{S}_2$, then $\mathcal{S} \cong \mathbb{C}$ or $\mathbb{C}^2$ and the result follows.

Extension

Recall $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = B_1 + iB_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Let $B_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $B_1,\ B_2,\ B_3$ are known as the **Pauley matrices**.

Conjecture 1 (Extension of Theorem 3a)

Suppose $A_1, A_2, A_3 \in \mathcal{B}(\mathcal{H})_{\rm sa}$. Then $W(A_1, A_2, A_3) \subseteq W(B_1, B_2, B_3)$ if and only if the map

$$\Phi(c_0 I_2 + c_1 B_1 + c_2 B_2 + c_3 B_3) = c_0 I_{\mathcal{H}} + c_1 A_1 + c_2 A_2 + c_3 A_3$$

is completely positive.

1) The conjecture fails for dim $\mathcal{H} = 2$. Just take $A_i = B_i^t$.

2) If dim $\mathcal{H} \neq 2$, $W(A_1, A_2, A_3)$ is convex but

$$W(B_1, B_2, B_3) = \{\mathbf{w} \in \mathbb{R}^3 : \|\mathbf{w}\| = 1\}.$$

If $W(A_1, A_2, A_3) \subseteq W(B_1, B_2, B_3)$, then $W(A_1, A_2, A_3)$ is a singleton. Therefore, all A_i are scalar and the conjecture holds.

Extension

Let $B_1,\ B_2,\ B_3$ be the Pauley matrices. Set $\hat{B}_i = B_i \oplus B_i^t$ for $i = 1, 2, 3$. Then

$$W(\hat{B}_1, \hat{B}_2, \hat{B}_3) = \{\mathbf{w} \in \mathbb{R}^3 : \|\mathbf{w}\| \leq 1\} \text{ is } \textbf{convex}.$$

Conjecture 2 (Extension of Theorem 3b)

Suppose $A_1, A_2, A_3 \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$. Let

$$\Phi(c_0 I_2 + c_1 \hat{B}_1 + c_2 \hat{B}_2 + c_3 \hat{B}_3) = c_0 I_{\mathcal{K}} + c_1 A_1 + c_2 A_2 + c_3 A_3$$

Then Φ is positive on $\mathrm{span}(I_2, \hat{B}_1, \hat{B}_2, \hat{B}_3)$ if and only if Φ is completely positive.

The conjecture holds if $\dim \mathcal{H} = n \leq 3$. Let $\Psi : M_2 \to M_n$ be given by $\Psi(X) = \Phi(X \oplus X^t)$. If Φ is positive and $n \leq 3$, then Ψ is **decomposable**. There exist completely positive $\Psi_1,\ \Psi_2 : M_2 \to M_n$ such that $\Psi(X) = \Psi_1(X) + \Psi_2(X)^t$. Then the result follows.

Question Does the result hold for all n?

Extension

Theorem 7 (Choi and Li)

Suppose $B \in M_2$ or $B = [b] \oplus B_1 \in M_3$. Then for all $A \in \mathcal{B}(\mathcal{H})$, $W(A) \subseteq W(B)$ if and only if the map

$$\Phi(c_0 I_2 + c_1 B + c_2 B^*) = c_0 I_{\mathcal{H}} + c_1 A + c_2 A^*$$

is completely positive.

Questions:

1) If $B \in M_3$ satisfies the conclusion in Theorem 7, must B be unitarily similar to $[b] \oplus B_1$?

2) For which subset S of $\mathbb{C}$ can we find $B \in M_n$ such that $W(B) = S$ and satisfies the conclusion in Theorem 7?

3) Does there exists $B \in M_n$ such that $W(B) =$ the square with vertices $\{1, -1, i, -i\}$ and satisfies the conclusion in Theorem 7?
$B = \operatorname{diag}(1, -1, i, -i)$ does not work.

Suppose $\mathcal{S}$ is an operator system of M_m and $k \geq 1$. For a fixed $\mathcal{H}$, let

$$P_k(\mathcal{S},\mathcal{H}) = \{\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H}) \text{ is } k\text{-positive}\}, \quad \text{and}$$

$$CP(\mathcal{S},\mathcal{H}) = \{\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H}) \text{ is completely posotive}\}.$$

Clearly,

$$CP(\mathcal{S},\mathcal{H}) \subset \cdots \subset P_k(\mathcal{S},\mathcal{H}) \subset \cdots \subset P_2(\mathcal{S},\mathcal{H}) \subset P_1(\mathcal{S},\mathcal{H})$$

The previous results shows that we have

$$CP(\mathcal{S},\mathcal{H}) = P_1(\mathcal{S},\mathcal{H})$$

for

1) $\mathcal{S} = \text{ span}(I, B, B^*)$ with $B \in M_2$ or $B = [b] \oplus B_1 \in M_3$ and any $\mathcal{H}$.

2) $\mathcal{S} = \text{ span}(I, B_1, B_2, B_3)$ and dim $\mathcal{H} \leq 3$.

Question: When will $CP(\mathcal{S},\mathcal{H}) = P_k(\mathcal{S},\mathcal{H})$?

Matrix range

Let $A \in \mathcal{B}(\mathcal{H})$. For each $m \geq 1$, Arveson defines the matrix range

$$\mathcal{W}_n(A) = \{\Phi(A) : \Phi \text{ is a unital completely positiive map from } \mathcal{B}(\mathcal{H}) \text{ to } M_n\}$$

Theorem 8 (Arveson)

1) $\mathcal{W}_n(A)$ is C* convex. That is, given $X_1, \ldots, X_k \in \mathcal{W}_n(A)$ and $Z_1, \ldots, Z_k \in M_n$ such that $\sum_{i=1}^k Z_i^* Z_i = I_n$, we have $\sum_{i=1}^k Z_i^* X_i Z_i \in \mathcal{W}_n(A)$. $\mathcal{W}_n(A)$ is the closure of the smallest C* convex set containing $W_n(A)$.

2) Let A be a normal operator and let $n \geq 1$. Then $\mathcal{W}_n(A)$ is the closure of

$$\{\sum_{i=1}^r \lambda_i H_i : r \geq 1,\ H_i \geq 0,\ \lambda_i \in \mathrm{sp}(T) \text{ and } \sum_{i=1}^r H_i = I_n\}$$

3) For some irreducible operators, the sequence $\{\mathcal{W}_n(A)\}_{n=1}^\infty$ is a complete invariant for unitary similarity.

Theorem 9 (Choi) Suppose $\Phi : M_n \to M_m$ is a linear map. Then the following conditions are equivalent:

(a) Φ is completely positive.

(b) Φ is k-positive for $k = \min(m, n)$.

(c) The Choi matrix $C(\Phi) = (\Phi(E_{ij}))$ is positive semidefinite.

(d) There exist $V_1, \ldots, V_r \in M_{n,\,m}$ such that

$$\Phi(A) = \sum_{j=1}^{r} V_j^* A V_j. \qquad (2)$$

Furthermore, suppose (d) holds. Then we have

(1) The map Φ is unital $(\Phi(I_n) = I_m)$ if and only if $\sum_{j=1}^{r} V_j^* V_j = I_m$.

(2) The map Φ is trace preserving $(\mathrm{tr}\,(\Phi(A)) = \mathrm{tr}\,(A))$ if and only if $\sum_{j=1}^{r} V_j V_j^* = I_n$.

The minimum of r in (2) is called the rank of Φ.

Joint matrix range

Given $n,\ m \geq 1$, let $CP(n,m)$ be the set of **unital completely positive maps** from M_n to M_m. For $1 \leq r \leq mn$, let $CP^r(n,m)$ be the set of $\Phi \in CP(n,m)$ of rank $\leq r$. Clearly,

$$CP^1(n,m) \subset CP^2(n,m) \subset \cdots \subset CP^{mn}(n,m) = CP(n,m)$$

Let $\mathbf{A} = (A_1,\ A_2, \ldots,\ A_p) \in H_n^p$. For each $m \geq 1$ and $1 \leq r \leq mn$, define

$$\mathcal{W}_m^r(\mathbf{A}) = \{(\Phi(A_1), \ldots, \Phi(A_p)) : \Phi \in CP^r(n,m)\}$$

We have

$$\mathcal{W}_m^1(\mathbf{A}) \subseteq \mathcal{W}_m^2(\mathbf{A}) \subseteq \cdots \subseteq \mathcal{W}_m^{mn}(\mathbf{A}) = \mathcal{W}_m(\mathbf{A})$$

Toeplitz-Haudorff Theorem: $\mathcal{W}_1^1(A_1, A_2) = \mathcal{W}_1(A_1, A_2)$.

Question: When will $\mathcal{W}_m^r(\mathbf{A}) = \mathcal{W}_m(\mathbf{A})$?

Note: $\mathcal{W}_m^r(A_1,\ A_2, \ldots,\ A_p) = \mathcal{W}_m^1(A_1 \otimes I_r,\ A_2 \otimes I_r, \ldots,\ A_p \otimes I_r)$.

Joint matrix range

Theorem 10

Suppose $A_1,\ A_2,\ldots,\ A_p \in H_n$. Let $1 \le r \le mn-1$. Then

$$\mathcal{W}_m^r(A_1,\ldots,A_p) = \mathcal{W}_m(A_1,\ldots,A_p) \quad (*)$$

if $m^2(p+1)-1 < (r+1)^2 - \delta_{mn,r+1}$.

For example, if $p = k^2-1$ and $n > k$, then

$$\mathcal{W}_m^{mk-1}(A_1,\ldots,A_p) = \mathcal{W}_m(A_1,\ldots,A_p)$$

for all $A_1,\ldots,A_p \in H_n$. In this case, one can show that $r = mk-1$ is the smallest number for (*) to hold. Putting $m = r = 1$, we have $\mathcal{W}_1^1(A_1,\ldots,A_p) = \mathcal{W}_1(A_1,\ldots,A_p)$ if

$$p < 2^2 - \delta_{n,2} = 4 - \delta_{n,2}.$$

Therefore, $W(A_1,A_2,A_3)$ is convex if $n \ge 3$.

Joint matrix range

Recall that for $\mathbf{A} = (A_1,\ A_2, \dots,\ A_p) \in H_n^p$,

$$\mathcal{W}_m^1(\mathbf{A}) \subseteq \mathcal{W}_m^2(\mathbf{A}) \subseteq \cdots \subseteq \mathcal{W}_m^{mn}(\mathbf{A}) = \mathcal{W}_m(\mathbf{A})$$

Let $\mathcal{S} = \text{span}(I_n, A_1,\ A_2, \dots,\ A_p)$ and $\mathcal{H} = \mathbb{C}^m$. Define

$$\mathcal{P}_k(\mathbf{A}) = \{(\Phi(A_1), \dots, \Phi(A_p)) : \Phi \in P_k(\mathcal{S}, \mathcal{H})\}$$

we have

$$\mathcal{W}_m(\mathbf{A}) \subseteq \cdots \subseteq \mathcal{P}_k(\mathbf{A}) \subseteq \cdots \subseteq \mathcal{P}_2(\mathbf{A}) \subseteq \mathcal{P}_1(\mathbf{A})$$

Note: $\mathcal{P}_k(\mathbf{A}) = \mathcal{W}_m(\mathbf{A}) \Leftrightarrow P_k(\mathcal{S}, M_m) = CP(\mathcal{S}, M_m)$

For $n \geq 3$, $p = 3$ we have

$$\mathcal{W}_1^1(\mathbf{A}) = \mathcal{P}_1(\mathbf{A})$$

Recall that for $A \in \mathcal{B}(\mathcal{H})$ and $m \le \dim \mathcal{H}$,

$$W_m(A) = \{B \in M_m : B \text{ is a compression of } A\}$$

Theorem 11 (Fan and Pall)

Suppose $A \in H_n$ has eigenvalues $a_1 \ge a_2 \ge \cdots \ge a_n$ and $1 \le m \le n$. Then $W_m(A)$ consists of all $B \in H_m$ with eigenvalues $b_1 \ge b_2 \ge \cdots \ge b_m$ satisfying the following inequalities:

$$a_i \ge b_i \ge a_{n-m+i} \qquad \text{for all } 1 \le i \le m$$

In particular, $B \in W_{n-1}(A)$ if and only if

$$a_1 \ge b_1 \ge a_2 \ge \cdots \ge b_{n-1} \ge a_n$$

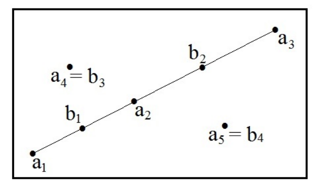
Suppose $A \in M_n$ is normal with eigenvalues $a_1, a_2, \ldots, a_n$. If $a_1, a_2, \ldots, a_n \in \mathbb{C}$ are collinear, then there exist $c \in \mathbb{C}$ and $\theta \in \mathbb{R}$ such that $e^{i\theta}A + cI_n \in H_n$ and we have

$$W_m(e^{i\theta}A + cI_n) = e^{i\theta}W_m(A) + cI_m$$

Compression of normal matrix

Theorem 12 (Fan and Pall)

Let $A \in M_n$ and $B \in M_{n-1}$ be normal matrices with eigenvalues $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_{n-1}$, respectively. Suppose $a_1, a_2, \ldots, a_q$ are each distinct from $b_1, b_2, \ldots, b_{q-1}$, while $a_i = b_{i-1}$ for $q+1 \leq i \leq n$. Then B is a compression of A if and only if $a_1, a_2, \ldots, a_q$ are collinear and every segment on this line limited by two adjacent a_i's contains one b_j, $1 \leq j \leq q-1$.



If no three a_i's are collinear, then up to permutation of indices, we must have $a_i = b_i$ for $i = 1, \ldots, n-2$ and $b_{n-1} \in \overline{a_{n-1}\ a_n}$.

Compression of normal matrix

Suppose $A \in M_n$ is normal with non-collinear eigenvalues $a_1, a_2, \ldots, a_n$. Let $\mathcal{D}_m(A) = \{\mathrm{diag}\,(B) : B \in W_m(A)\} \subset \mathbb{C}^m$.

Theorem 13

Suppose $A \in M_n$ is normal with non-collinear eigenvalues $a_1, a_2, \ldots, a_n$. Then the following conditions are equivalent:

1) $\mathcal{D}_m(A)$ is convex.

2) $W_m(A)$ is convex.

3) $W_m(A)$ is C*-convex. ($\Leftrightarrow \mathcal{W}_m^1(A) = W_m(A) = \mathcal{W}_m(A)$)

4) Every vertex of $W(A)$ has multiplicity $\geq m$.

Common compression of matrices

Given $A \in M_n$, $B \in M_m$ and $1 \leq k \leq n,\ m$. A and B is said to have a **common k-dimensional compression** if there exist $U \in M_{m,k}$ and $V \in M_{m,k}$ such that $U^*U = I_k = V^*V$ and $U^*AU = V^*BV$.

For $m = k \leq n$, this is equivalent to the compression of Hermitian matrices studied by Fan and Pall.

Extension of the result of Fan and Pall

Theorem 14 Suppose and $A \in H_n$ and $B \in H_m$ have eigenvalues $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_m$, respectively, and $1 \leq k \leq n,\ m$. Then B and C have a common k-dimensional compression if and only if the following inequalities hold:

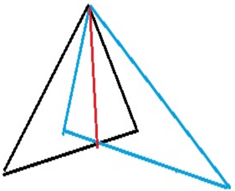
$$a_i \geq b_{m-k+i} \text{ and } b_i \geq a_{n-k+i} \qquad \text{for all } 1 \leq i \leq k$$

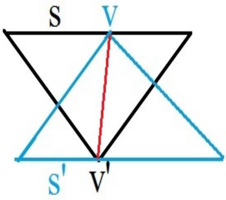
Theorem 15 Suppose A and B are 3×3 normal matrices with non collinear eigenvalues $a_1,\ a_2, a_3$ and $b_1,\ b_2, b_3$ respectively. Then

1) A and B have a common 2-dimensional compression C, with degenerate $W(C)$ ($\Leftrightarrow$ C is normal), if and only if either

(a) $W(A)$ and $W(B)$ have a vertex in common and the corresponding opposite sides intersect, **or**

(b) one vertex v of $W(A)$ lies on an edge s of $W(B)$ and the vertex v' in $W(B)$ opposite to s lies on the edge s' in $W(A)$ opposite to v.



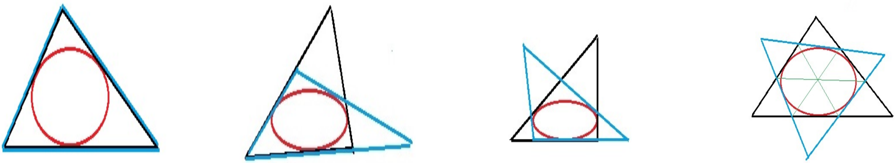


2) A and B have a common 2-dimensional compression C, with non-degenerate $W(C)$ ($\Leftrightarrow$ C is not normal), if and only if the following conditions are satisfied:

(a) $W(A) \cap W(B)$ is an m-sided polygon P with $m \geq 3$.

(b) Every edge of $W(A)$ and $W(B)$ intersects a side of P at more than one point.

(c) For $m = 6$, the diagonals of P are concurrent.



$m = 3$ $m = 4$ $m = 5$ $m = 6$