# Unique Pseudo-Expectations, Dynamics, and Minimal Norms

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# Some Motivation

Consider

- $\mathcal{D}$ a (unital) abelian $C^*$-algebra,
- $\Gamma$ a discrete group, and
- $\Gamma \ni t \mapsto \alpha_t \in \text{Aut}(\mathcal{D})$, action of $\Gamma$ on $\mathcal{D}$.

Let $\mathcal{A}_0 := C_c(\Gamma, \mathcal{D})$ be the (twisted) $*$-algebra of all finitely supported functions $h : \Gamma \to \mathcal{D}$.
In general, there are many $C^*$-norms on $\mathcal{A}_0$.

**Question**

*Is there a minimal $C^*$ norm on $\mathcal{A}_0$?*

**Answer**

*Not usually.*

# An Example

The answer is no even in very elementary cases.

## Example

Consider $X = \{1\}$ and $\Gamma = \mathbb{Z}$. Then $\mathcal{A}_0$ is set of trig polys. If $\eta$ a $C^*$-norm on $\mathcal{A}_0$, $\exists\, K \subseteq \mathbb{T}$ compact with $\operatorname{card}(K) = \infty$ such that

$$\eta(h) = \sup_{w \in K} \Big| \sum_{n \in \mathbb{Z}} h(n) w^n \Big| \qquad (h \in \mathcal{A}_0).$$

If $K_1 \subsetneq K$ is compact and $\operatorname{card}(K_1) = \infty$, then

$$\eta_1(h) := \sup_{w \in K_1} \Big| \sum_{n \in \mathbb{Z}} h(n) w^n \Big|$$

is a $C^*$-norm with $\eta_1(h) \leq \eta(h)$ and $\eta_1 \neq \eta$.

# A Setting With a Minimal Norm

However, in some cases there is a minimal norm.
Recall:

- the action of $\Gamma$ dualizes to an action of $\Gamma$ on the Gelfand space $\hat{\mathcal{D}}$: for $t \in \Gamma$, $\hat{\mathcal{D}} \ni \sigma \mapsto \sigma \circ \alpha_t$.
- $\Gamma$ acts *topologically freely* on $X$ if $\forall\, t \in \Gamma \setminus \{e\}$, $\operatorname{int}\{x \in X : tx = x\} = \emptyset$.

**Fact (Corollary of Theorem B Below)**

*Suppose $\Gamma$ acts topologically freely on $\hat{\mathcal{D}}$ and let $\|\cdot\|_{red}$ be the reduced crossed product norm on $C_c(\Gamma, \mathcal{D})$.*
*If $\eta$ is any $C^*$-norm on $C_c(\Gamma, \mathcal{D})$, then $\forall\, h \in C_c(\Gamma, \mathcal{D})$*

$$\|h\|_{red} \leq \eta(h).$$

# A Related Result of Rainone on Crossed Prod's

C. Schafhauser alerted me to the following result:

**Proposition (Rainone)**

*Let $\Gamma$ be a discrete gp. acting on a $C^*$-algebra $\mathfrak{A}$. If $\eta$ is a norm on $C_c(\Gamma, \mathfrak{A})$ such that the cannonical conditional expectation,*

$$C_c(\Gamma, \mathfrak{A}) \ni f \mapsto f(1) \in \mathfrak{A}$$

*is $\eta$-bounded, then $\|f\|_{red} \leq \eta(f)$ for all $f \in C_c(\Gamma, \mathfrak{A})$.*

It's not always clear how to verify this hypothesis, so we'll go in another direction.

# A General Context: Regular Inclusions

## Definitions

- An *inclusion* is a pair $(\mathcal{C}, \mathcal{D})$ of unital $C^*$-algebras (with same unit) and $\mathcal{D}$ abelian
- An inclusion is *regular* if

$$\mathcal{N}(\mathcal{C}, \mathcal{D}) := \{v \in \mathcal{C} : v\mathcal{D}v^* \cup v^*\mathcal{D}v \subseteq \mathcal{D}\}$$

  has dense span in $\mathcal{C}$. Elements of $\mathcal{N}(\mathcal{C}, \mathcal{D})$ are *normalizers*.
- If $\mathcal{D}$ is a MASA in $\mathcal{C}$, call $(\mathcal{C}, \mathcal{D})$ a MASA inclusion.

## Example

If $(\mathcal{D}, \Gamma)$ a $C^*$-dyn. system with $\mathcal{D}$ abelian & $\Gamma$ acts top. freely on $\hat{\mathcal{D}}$, then $(\mathcal{D} \rtimes_{red} \Gamma, \mathcal{D})$ is a regular MASA inclusion.

# Regular MASA Inclusions appearing in Literature

Certain regular MASA inclusions have been studied:

Cartan Inclusions: A reg. MASA inclusion $(\mathcal{C}, \mathcal{D})$ is a Cartan inclusion if there exists a faithful cond. expect. $E : \mathcal{C} \to \mathcal{D}$. Defined by Renault; intended to be the $C^*$-analog of a Cartan MASA in a von Neumann alg.

$C^*$ Diagonals: An incl. $(\mathcal{C}, \mathcal{D})$ is a $C^*$-diagonal if it is Cartan & every pure state on $\mathcal{D}$ extends uniquely to state on $\mathcal{C}$.
Introduced by Kumjian; have very nice properties.

# Some Examples of Cartan & $C^*$-Diagonals

## Examples

- $(M_n(\mathbb{C}), D_n)$ (the prototype example of a $C^*$-diag.)
- $(C(\mathbb{T}) \rtimes \mathbb{Z}, C(\mathbb{T}))$, where action is irrational rotation is a $C^*$-diag;
- Let $1 < n \in \mathbb{N} \cup \{\infty\}$, $S := (S_1, \dots, S_n)$ be isometries generating $\mathcal{O}_n$ and let $\mathcal{D} := \overline{\text{span}}\{ww^* : w \in \{S_{i_1} \cdots S_{i_k}\}\}$. Then $(\mathcal{O}_n, \mathcal{D})$ is Cartan, but not a $C^*$-diag.

# A Side Problem

**FACT** (Archbold-Bunce-Gregson): Whenever $(\mathcal{C}, \mathcal{D})$ is an inclusion with the extension property (an EP-inclusion), $\mathcal{D}$ is a MASA in $\mathcal{C}$ & $\exists!$ conditional expectation $E : \mathcal{C} \to \mathcal{D}$.

**Theorem (Donsig-P., JOT 2007)**

*Let $(\mathcal{C}, \mathcal{D})$ be a regular EP-inclusion with cond. expect. $E$. Then the left kernel $\mathcal{L} := \{x \in \mathcal{C} : E(x^*x) = 0\}$ is an ideal, $\mathcal{L} \cap \mathcal{D} = (0)$ and $(\mathcal{C}/\mathcal{L}, \mathcal{D})$ is a $C^*$-diagonal.*

The definition of $C^*$-diagonal leads to the following question:

**Irritating Side Problem**

*Give an example of a regular EP-inclusion which isn't a $C^*$-diag (i.e. with non-faithful C.E.).*

# Lack of Conditional Expectation

The most studied reg. inclusions have a cond. expect., which is a very useful tool in their analysis.

A general reg. MASA inclusion $(\mathcal{C}, \mathcal{D})$ can fail to have a cond. expect. $E : \mathcal{C} \to \mathcal{D}$.

**Example**

Let $X := \{z \in \mathbb{C} : \mathrm{Re}(z)\mathrm{Im}(z) = 0 \,\&\, |z| \leq 1\}$. $\mathbb{Z}_2$ acts on $X$ via $z \mapsto \overline{z}$. Put $\mathcal{C} := C(X) \rtimes \mathbb{Z}_2$ and $\mathcal{D} := C(X)^c$ (rel. commutant).

Easy computations show:

- $(\mathcal{C}, \mathcal{D})$ is a reg. MASA inclusion, but $\nexists$ a C.E. $E : \mathcal{C} \to \mathcal{D}$.

We'll need a replacement for conditional expectations.

# Injective Envelopes & The Dixmier Algebra

For an abelian $C^*$-algebra $\mathcal{D}$, $(I(\mathcal{D}), \iota)$ is an *injective envelope* for $\mathcal{D}$, if

- $I(\mathcal{D})$ an injective $C^*$-algebra,
- $\iota : \mathcal{D} \to I(\mathcal{D})$ a $*$-monomorphism; &
- if $J \subseteq I(\mathcal{D})$ an ideal with $J \cap \iota(\mathcal{D}) = (0)$, then $J = (0)$.

When $\mathcal{D} = C(X)$, the Dixmier algebra is

$$\mathrm{Dix}(X) := \{\text{Bounded Borel Ftns on } X\}/\mathcal{N},$$

where $\mathcal{N} = \{f \text{ bdd, Borel} : \{x \in X : |f(x)| \neq 0\} \text{ is meager.}\}$

**Theorem (Dixmier)**

*$(\mathrm{Dix}(X), \iota)$, where $C(X) \ni f \mapsto \iota(f) = f + \mathcal{N}$, "is" the injective envelope for $C(X)$.*

# A Replacement for Conditional Expectation

**Definition**

Let $(\mathcal{C}, \mathcal{D})$ be an inclusion, & $(I(\mathcal{D}), \iota)$ an inj. envelope for $\mathcal{D}$. A pseudo-expectation for $(\mathcal{C}, \mathcal{D})$ is a completely positive unital map $E : \mathcal{C} \to I(\mathcal{D})$ such that $E|_{\mathcal{D}} = \iota$.

THE INJECTIVITY OF $I(\mathcal{D})$ ENSURES EXISTENCE OF $E$.

In general, there are many pseudo-expectations: e.g. every state on $\mathcal{C}$ is a pseudo-expecation for $(\mathcal{C}, \mathbb{C}I)$.

However, ...

# Properties of Regular MASA Inclusions

Regular MASA inclusions have unique pseudo-expectations.

**Theorem A (D.P. 2012)**

*If* $(\mathcal{C}, \mathcal{D})$ *a regular MASA inclusion, then* $\exists!$ *pseudo-expectation* $E : \mathcal{C} \to I(\mathcal{D})$ *and*

$$\mathcal{L}(\mathcal{C}, \mathcal{D}) := \{x \in \mathcal{C} : E(x^*x) = 0\}$$

*is a (closed) 2-sided ideal in* $\mathcal{C}$ *with* $\mathcal{L}(\mathcal{C}, \mathcal{D}) \cap \mathcal{D} = (0)$*.*
*Also, if* $J \subseteq \mathcal{C}$ *a closed ideal with* $J \cap \mathcal{D} = (0)$*, then* $J \subseteq \mathcal{L}(\mathcal{C}, \mathcal{D})$*.*

**Note:** When $\exists$ a conditional expectation of $\mathcal{C}$ onto $\mathcal{D}$, it is the pseudo-expectation.

# A Minimal Seminorm

**Definition**

A *skeleton* for the inclusion $(\mathcal{C}, \mathcal{D})$ is a $*$-submonoid $\mathcal{M} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$ s.t.

$$\mathcal{D} \subseteq \operatorname{span} \mathcal{M} \quad \text{and} \quad \overline{\operatorname{span}} \mathcal{M} = \mathcal{C}.$$

Note: $\operatorname{span} \mathcal{M}$ is a dense $*$-subalgebra of $\mathcal{C}$.

**Example:** For a $C^*$-dynam. sys. $(\mathcal{D}, \Gamma)$, $\{d\delta_t : d \in \mathcal{D}, t \in \Gamma\}$ is a skeleton for $(\mathcal{D} \rtimes_{red} \Gamma, \mathcal{D})$.

**Theorem B**

*Suppose* $\mathcal{M}$ *is a skeleton for the reg. MASA inclusion* $(\mathcal{C}, \mathcal{D})$*. For any* $C^*$*-norm* $\eta$ *on* $\operatorname{span} \mathcal{M}$*,*

$$\operatorname{dist}(x, \mathcal{L}(\mathcal{C}, \mathcal{D})) \leq \eta(x) \quad \forall x \in \operatorname{span} \mathcal{M}.$$

# Outline of Proof of Theorem B

For any $C^*$-norm $\eta$ on $\operatorname{span}\mathcal{M}$, let $\mathcal{C}_\eta$ be completion, so $(\mathcal{C}_\eta, \mathcal{D})$ an inclusion.

- Show $\exists!$ pseudo-expectation $E_\eta : \mathcal{C}_\eta \to I(\mathcal{D})$ and
$$E_\eta|_{\operatorname{span}\mathcal{M}} = E|_{\operatorname{span}\mathcal{M}}.$$
Proof is similar to showing uniqueness of $E : \mathcal{C} \to I(\mathcal{D})$. (More on this later.)
- For $x \in \operatorname{span}\mathcal{M}$, $\operatorname{dist}(x, \mathcal{L}(\mathcal{C}, \mathcal{D})) = \|\pi_E(x)\|$, where $\pi_E$ is Steinspring rep'n for $E$.
- Finally, for $x \in \operatorname{span}\mathcal{M}$,
$$\operatorname{dist}(x, \mathcal{L}(\mathcal{C}, \mathcal{D})) = \|\pi_E(x)\| = \left\|\pi_{E_\eta}(x)\right\| \leq \eta(x).$$

# Virtual Cartan Inclusions

A virtual Cartan inclusion is a reg. MASA incl'n such that $\mathcal{L}(\mathcal{C}, \mathcal{D}) = (0)$ (i.e. $E$ faithful).
Virtual Cartan inclusions have a uniqueness property:

**Fact**

*Let $(\mathcal{C}, \mathcal{D})$ be a regular MASA incl'n. TFAE:*

1. $(\mathcal{C}, \mathcal{D})$ *a virtual Cartan incl'n;*
2. *whenever* $\pi : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ *is a rep'n and* $\pi|_{\mathcal{D}}$ *is faithful, then* $\pi$ *is faithful on* $\mathcal{C}$.

Every Cartan incl'n & every $C^*$-diag is a virtual Cartan incl'n.
(Also various graph algebras are virtual Cartan inclusions.)

# Some Nice Features of Virtual Cartan Inclusions

**Theorem**

*If $(\mathcal{C}, \mathcal{D})$ is vir. Cartan, then $\mathcal{D}$ norms $\mathcal{C}$.*

Unique faithful pseudo-expectation leads to:

**Theorem**

*Let $(\mathcal{C}, \mathcal{D})$ be a virtual Cartan incl.. If $\mathcal{A} \subseteq \mathcal{C}$ is a closed subalgebra (not nec. $*$) with $\mathcal{D} \subseteq \mathcal{A}$, then*

$$C^*_{env}(\mathcal{A}) = C^*(\mathcal{A}) \subseteq \mathcal{C}.$$

**Theorem**

*Suppose $(\mathcal{C}_i, \mathcal{D}_i)$ are vir. Cartan & $\mathcal{A}_i \subseteq \mathcal{C}_i$ are subalg's s.t. $\mathcal{D}_i \subseteq \mathcal{A}_i$. If $u : \mathcal{A}_1 \to A_2$ is an isometric isomorphism, $\exists!$ $*$-iso $\tilde{u} : C^*(\mathcal{A}_1) \to C^*(\mathcal{A}_2)$ extending $u$.*

Let $(\mathcal{C}, \mathcal{D})$ be a regular MASA inclusion. Recall that $\mathcal{D} \cap \mathcal{L}(\mathcal{C}, \mathcal{D}) = (0)$, so $(\mathcal{C}/\mathcal{L}(\mathcal{C}, \mathcal{D}), \mathcal{D})$ is a regular inclusion.

Unclear if $\mathcal{D}$ a MASA in $\mathcal{C}/\mathcal{L}(\mathcal{C}, \mathcal{D})$. But letting $\mathcal{D}^c$ be relative commutant of $\mathcal{D}$ in $\mathcal{C}/\mathcal{L}(\mathcal{C}, \mathcal{D})$, get

**Theorem**

*Suppose* $(\mathcal{C}, \mathcal{D})$ *a regular MASA inclusion. Then*

1. $\mathcal{D}^c$ *is abelian, &* $(\mathcal{C}/\mathcal{L}(\mathcal{C}, \mathcal{D}), \mathcal{D}^c)$ *is a virtual Cartan inclusion.*
2. *If* $\exists$ *a cond. expect.* $E : \mathcal{C} \to \mathcal{D}$*, then* $\mathcal{D} = \mathcal{D}^c$ *&* $(\mathcal{C}/\mathcal{L}(\mathcal{C}, \mathcal{D}), \mathcal{D})$ *is a Cartan inclusion.*

# Maximal & Minimal Norms on Virtual Cartan Inclusions

## Corollary of Theorem B

*If* $(\mathcal{C}, \mathcal{D})$ *is a virtual Cartan incl'n and* $\mathcal{M}$ *is a skeleton, then* $\|\|$ *is the minimal* $C^*$*-norm on* $\operatorname{span}\mathcal{M}$*.*
*Moreover, there exists a maximal* $C^*$*-norm* $\|\cdot\|_{max}$ *on* $\operatorname{span}\mathcal{M}$*.*

Tempting to say that the virtual Cartan incl'n $(\mathcal{C}, \mathcal{D})$ is amenable if $\|\cdot\|_{min} = \|\cdot\|_{max}$:

## Question

*The family* $\{\theta_{v^*} : v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\}$ *is an inverse semigroup acting as partial automorphisms of* $\mathcal{D}$*. Is there a notion of amenable action for inverse semigroups which ensures that* $\|\cdot\|_{min} = \|\cdot\|_{max}$ *precisely when the action is amenable?*

# Application to Dynamical Systems

Consider the reduced crossed prod. $\mathcal{D} \rtimes_{red} \Gamma$ where $\mathcal{D}$ abelian & $\Gamma$ discrete.

**Theorem (Pitts, '12)**

*$(\mathcal{D} \rtimes_{red} \Gamma, \mathcal{D})$ is a virtual Cartan inclusion iff $\forall \sigma \in \hat{\mathcal{D}}$, the germ isotropy group*

$$H^\sigma := \{s \in \Gamma : \sigma \in (Fix(s))^\circ\}$$

*is abelian.*

So, if $\Gamma$ acts topologically freely on $\hat{\mathcal{D}}$, then $\forall \sigma \in \hat{\mathcal{D}}$, $H^\sigma = \{e\}$.

**Corollary**

*If $H^\sigma$ is abelian for all $\sigma \in \hat{\mathcal{D}}$, then the reduced crossed product norm is the smallest $C^*$-norm on $\operatorname{span}\{d\delta_t : t \in \Gamma, d \in \mathcal{D}\}$.*

# Another Application: Unique Extensions

Recall $(\mathcal{C}, \mathcal{D})$ has extension property (EP) if every $\sigma \in \hat{\mathcal{D}}$ extends uniquely to $\tilde{\sigma} \in \text{State}(\mathcal{C})$.
Quotients inherit the EP:

**Fact (Archbold-Bunce-Gregson)**

*If $(\mathcal{C}, \mathcal{D})$ is an EP-inclusion, & $J \subseteq \mathcal{C}$ is an ideal, then $(\mathcal{C}/J,\ \mathcal{D}/(\mathcal{D} \cap J))$ is EP.*

We can go the other way too:

**Theorem**

*Let $\mathcal{M}$ be a skeleton for the reg. MASA incl'n $(\mathcal{C}, \mathcal{D})$ & let $\eta$ be a $C^*$-norm on $\operatorname{span} \mathcal{M}$. If $(\mathcal{C}, \mathcal{D})$ has EP, so does $(\mathcal{C}_\eta, \mathcal{D})$.*

Theorem holds for $C^*$-seminorms too.

# Uniqueness of Pseudo-expecations

We now discuss the key ideas in the proof of

**Theorem A**

*If* $(\mathcal{C}, \mathcal{D})$ *a regular MASA inclusion, then* $\exists!$ *pseudo-expectation* $E : \mathcal{C} \to I(\mathcal{D})$ *and*

$$\mathcal{L}(\mathcal{C}, \mathcal{D}) := \{x \in \mathcal{C} : E(x^*x) = 0\}$$

*is a (closed) 2-sided ideal in* $\mathcal{C}$ *with* $\mathcal{L}(\mathcal{C}, \mathcal{D}) \cap \mathcal{D} = (0)$*.*
*Also, if* $J \subseteq \mathcal{C}$ *a closed ideal with* $J \cap \mathcal{D} = (0)$*, then* $J \subseteq \mathcal{L}(\mathcal{C}, \mathcal{D})$*.*

The ideas highlight relationship between partial actions on $\mathcal{D}$ and properties of $I(\mathcal{D})$.

# Some Dynamics for Regular Inclusions

## Fact

*For an inclusion* $(\mathcal{C}, \mathcal{D})$ *and* $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, *the map* $vv^*d \mapsto v^*dv$ *extends uniquely to a* $*$*-isomorphism* $\theta_v : \overline{vv^*\mathcal{D}} \to \overline{v^*v\mathcal{D}}$ *&*

$$v\theta_v(h) = hv \quad \forall\, h \in \overline{vv^*\mathcal{D}}$$

## Extending Isomorphisms of Ideals of $\mathcal{D}$ to $I(\mathcal{D})$

*Let* $\mathcal{D}$ *be an abelian* $C^*$*-algebra. For* $i = 1, 2$, *let* $J_i \lhd \mathcal{D}$, *& let*

$$P_i = \sup_{I(\mathcal{D})}(\textit{a. u. for } J_i) \in \mathrm{PROJ}(I(\mathcal{D}))$$

*be "support proj" for* $J_i$.

*If* $\theta : J_1 \to J_2$ *an isomorphism,* $\exists!$ *isomorphism*

$$\tilde{\theta} : P_1 I(\mathcal{D}) \to P_2 I(\mathcal{D})$$

*extending* $\theta$ ($\tilde{\theta} \circ \iota = \iota \circ \theta$).

# Frolík's Theorem

**Theorem (Frolík)**

*If* $\mathfrak{I}$ *injective, abelian,* $C^*$*-algebra,* $P, Q \in \text{PROJ}(\mathfrak{I})$ *&* $\alpha : P\mathfrak{I} \to Q\mathfrak{I}$ *is a* $*$*-iso. Then* $\exists\ \{R_i\}_{i=0}^3 \subseteq \text{PROJ}\,\mathfrak{I}$ *s.t.*

1. $P = \sum_{j=0}^3 R_i$;
2. $\alpha|_{R_0\mathfrak{I}} = id|_{R_0\mathfrak{I}}$; and
3. for $i = 1, 2, 3$, $R_i\alpha(R_i) = 0$.

Note: $R_0$ corresponds to fixed points for $\alpha$ and other $R_i$ are "free parts" of $\alpha$.

# Frolík Decomposition for $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$: Motivation

Given $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, let $P$, $Q$ be support proj's for $\overline{vv^*\mathcal{D}}$ & $\overline{v^*v\mathcal{D}}$.
Apply Frolík to $\tilde{\theta}_v : PI(\mathcal{D}) \to QI(\mathcal{D})$, get $\{R_i\}_{i=0}^3$.
If we could write,

$$v = R_0 v + R_1 v + R_2 v + R_3 v, \tag{1}$$

then for any pseudo-expect, $E$,

$$\begin{aligned} E(v) &= E(R_0 v) + E(R_1 v) + E(R_2 v) + E(R_3 v) \\ &= E(R_0 v) + E(R_1 v R_1) + E(R_2 v R_2) + E(R_3 v R_3) \\ &= E(R_0 v) + \sum_{i=1}^{3} E(v \tilde{\theta}_v(R_i) R_i) = E(R_0 v). \end{aligned}$$

But products in (1) not defined!

# Frolík Decomposition

Put

$$K_i := \{d \in \mathcal{D} : \iota(d)R_i = \iota(d)\} \quad (i = 0, \dots, 3)$$
$$K_4 := \{d \in \mathcal{D} : vv^*d = 0\}.$$

Then $K_i$ pairwise disjoint closed ideals in $\mathcal{D}$ and

1. $K := \bigvee_{i=0}^{4} K_i$ an essential ideal in $\mathcal{D}$;
2. for $i = 1, 2, 3, 4$, & $h, k \in K_i$, $hvk = 0$;
3. for $d \in K_0$, $dv = vd \in \mathcal{D}$ (requires $(\mathcal{C}, \mathcal{D})$ a MASA incl'n).

So instead of (1) we think of $v$ decomposed as

$$v \text{ "=" } K_0 v + K_1 v + K_2 v + K_3 v.$$

# Uniqueness of Pseudo-Expectations

$(\mathcal{C}, \mathcal{D})$ a reg. MASA incl., $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$
Let $E_1, E_2$ be pseudo-expectations,

$K_i v K_i = 0, (i = 1, \dots, 4) \Rightarrow E_i(vK_i) = 0;$
$dv = vd \in \mathcal{D} \,\forall\, d \in K_0 \Rightarrow E_1(vd) = E_2(vd), d \in K_0.$

So $E_1 = E_2$ on $\cup_{i=1}^4 K_i$ & finally
$\bigvee_0^4 K_i$ essential $\Rightarrow E_1(v) = E_2(v).$

As span $\mathcal{N}(\mathcal{C}, \mathcal{D})$ is dense, $E_1 = E_2$.

# Why $\mathcal{L}(\mathcal{C}, \mathcal{D})$ is a Right Ideal

Cauchy-Schwartz for ucp maps gives $\mathcal{L}(\mathcal{C}, \mathcal{D})$ is a left ideal.

If $(\mathcal{C}, \mathcal{D})$ has extension property, easy to show that when $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ & $x \in \mathcal{C}$, $E(v^*xv) = v^*E(x)v$.

For a reg. MASA incl, $(\mathcal{C}, \mathcal{D})$, the products on right aren't def'nd. Rewrite this using $\theta_v$: Get

$$E(v^*xv) = \theta_v(vv^*E(x)) = \theta_v(E(vv^*x)).$$

Using Frolík ideals & regularity, can show:

$$E(v^*xv) = \tilde{\theta}_v(E(vv^*x))\, \forall x \in \mathcal{C}.$$

So for $y \in \mathcal{L}(\mathcal{C}, \mathcal{D})$, $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$,

$$E(v^*y^*yv) = \tilde{\theta}_v(E(vv^*y^*y)) = \tilde{\theta}_v(\iota(vv^*)E(y^*y)) = 0,$$

so $yv \in \mathcal{L}(\mathcal{C}, \mathcal{D})$. Then regularity gives $\mathcal{L}(C, \mathcal{D})$ right-ideal.

# Three Bonuses from Frolík Decompositions

The ideas involved with Frolík decompositions can be used to produce the following results.

**Bonus 1**

*Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion with $\mathcal{D}$ injective. Then $(\mathcal{C}, \mathcal{D})$ is an EP-inclusion $\Leftrightarrow$ $(\mathcal{C}, \mathcal{D})$ is a MASA inclusion.*

**Bonus 2**

*If $(\mathcal{C}, \mathcal{D})$ is a regular (or skeletal) MASA inclusion with $\mathcal{D}$ injective, then $\mathcal{L}(\mathcal{C}, \mathcal{D}) \cap \operatorname{span} \mathcal{N}(\mathcal{C}, \mathcal{D}) = (0)$.*

Let $(X, \Gamma)$ be a (discrete) dynam. system, $(P, f)$ a projective cover for $X$ (corresponds to injective envelope of $C(X)$).

**Bonus 3**

1. *The action of* $\Gamma$ *uniquely "lifts" to produce a dynam. system* $(P, \Gamma)$ *with* $f(s \cdot p) = s \cdot f(p)$ *(*$p \in P, s \in \Gamma$*); and*
2. $(X, \Gamma)$ *is topologically free* $\Leftrightarrow$ $(P, \Gamma)$ *is free.*

Part (1) is known (e.g. Hadwin-Paulsen), but is part (2) known?

# Answering the Irritating Side Problem

**Irritating Side Problem**

*Find example of a regular EP-inclusion with non-faithful C.E..*

Let

- $\mathcal{H}$ a Hilbert space with $\dim \mathcal{H} = \aleph_0$;
- $\mathcal{D}$ a non-atomic MASA in $\mathcal{B}(\mathcal{H})$; and
- $\mathcal{C} = \overline{\text{span}}^{\|\ \|} \mathcal{N}(\mathcal{B}(\mathcal{H}), \mathcal{D})$.

Then $\mathcal{D}$ a MASA in $\mathcal{C}$ & $\mathcal{D}$ injective. Bonus 1 gives $(\mathcal{C}, \mathcal{D})$ a regular EP inclusion. (Note: $(\mathcal{B}(\mathcal{H}), \mathcal{D})$ doesn't have EP!) So:

**Question (P.)**

*Is* $E : \mathcal{C} \to \mathcal{D}$ *faithful?*

Here's a (slight) modification & special case of their answer.
Let $\Gamma = SL_3(\mathbb{Z}) \leftarrow$ has property (T). Action of $\Gamma$ on $\mathbb{R}^3$ induces action of $\Gamma$ on $(\mathbb{T}^3, Haar)$ which is

- meas. preserving & ergodic.
- Put $\mathcal{H} = L^2(\mathbb{T}^3)$ & $\mathcal{D} := \{M_f : f \in L^\infty(\mathbb{T}^3)\}$.
- Get unitary rep'n: $s \mapsto U_s$, where $U_s\xi = \xi \circ s^{-1}$.

Then $U_s \in \mathcal{N}(\mathcal{B}(\mathcal{H}), \mathcal{D})$, $s \in \Gamma$.

**Key Observation (Johnson & Zarikian)**

*Property (T) & a 1985 theorem of Chou, Lau, Rosenblatt give*

$$Proj_{\mathbb{C}\mathbf{1}} \in C^*(\{U_s\}_{s\in\Gamma}).$$

As $\mathcal{C}$ is irreducible, $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{C}$. But $\mathcal{K}(\mathcal{H}) \subseteq \ker E$, so $E$ not faithful.

# More on this Example

**Remark**

*Bonus 2 gives*

$$\text{span}(\mathcal{N}(\mathcal{B}(\mathcal{H}), \mathcal{D})) \cap \mathcal{K}(\mathcal{H}) = (0),$$

*even though*

$$\mathcal{K}(\mathcal{H}) \subseteq \overline{\text{span}}^{\| \|}(\mathcal{N}(\mathcal{B}(\mathcal{H}), \mathcal{D})).$$

THANK YOU!