Euclidean Scattering and the Problem of Moments

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Outline

Historical Perspective

- Einstein's Special Theory of Relativity is formulated on a vector space $M = \mathbb{R}^{1,3}$, which is supplied with a nondegenerate, symmetric bilinear form defined by

$$\langle x, y \rangle_M = -x^0 y^0 + \mathbf{x} \cdot \mathbf{y}$$

- Vectors in M are often referred to as either: TL, SL, or LL depending on whether the associated quadratic form

$$Q(x) = \langle x, x \rangle_M$$

satisfies: $Q(x) < 0$, $Q(x) > 0$, or $Q(x) = 0$.

Historical Perspective

- A linear map $\Lambda : M \to M$ that preserves the Minkowski form

$$\langle \Lambda x, \Lambda y \rangle_M = \langle x, y \rangle_M \qquad \forall x, y \in M$$

is known as a Lorentz transformation.

- The Lorentz transformations form a six dimensional non-compact, non-abelian, non-connected Lie group.

Historical Perspective

- The non-relativistic Schrödinger picture of quantum mechanics is described by time-dependent state vectors $\psi(x)$ belonging to a Hilbert space.

- Time-evolution of the quantum states is governed by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x) = \hat{H}\psi(x), \qquad \text{with} \quad \hat{H} = -\Delta + \hat{V}$$

  where $\hat{H}$ – the Hamiltonian – is a self-adjoint operator which serves as the generator of time-translation for the system.

Historical Perspective

- Despite its success, the Schrödinger equation possesses several critical shortcomings.

- Perhaps most importantly, the Schrödinger equation is not Lorentz-invariant; i.e.

$$x \mapsto x' = \Lambda x \not\Rightarrow i\hbar \frac{\partial}{\partial t'} \psi'(x') = \hat{H}' \psi'(x')$$

  and so it cannot be used to describe relativistic particles – e.g. photons.

Historical Perspective

- Quantum field theory was born out of an effort to marry the theory of relativity with quantum mechanics.

- The business of quantum field theory is to write down fields – functions defined over spacetime – which also possess the discrete behavior characteristic of quantum mechanics.

- The first field to be successfully quantized was the electromagnetic field in the absence of sources, and, according to Weinberg (2005), "is still the paradigmatic example of a successful quantum field theory."

The Wightman Axioms

- In light of the success of quantum electrodynamics (QED), there were attempts to develop the theory in a mathematically rigorous way in an effort to encompass other types of quantum fields.

- In the 1950's, Arthur S. Wightman gave a precise mathematical definition for a quantum field theory by listing the properties they should have.

The Wightman Axioms

The ingredients for the Wightman axioms are a Hilbert space $\mathcal{H}$ equipped with:

1. A dense subspace $\mathcal{D}$.
2. An operator-valued distribution ϕ on $\mathbb{R}^4$.
3. A unitary representation U of the Poincaré group on $\mathcal{H}$.
4. A unique state $\psi_0 \in \mathcal{H}$ such that $U(a, \Lambda)\psi_0 = \psi_0$ for all $a \in \mathbb{R}^4$ and all $\Lambda \in$ Lorentz Group.

The Reconstruction Theorem

The precise nature of the Wightman axioms is immaterial for our purposes. What is important, however, is to note that constructing non-trivial field theories (i.e. field theories for interacting particles) is highly non-trivial.

Following a path illuminated by Julian Schwinger, Konrad Osterwalder and Robert Schrader provided a set of axioms for a collection of objects S_n known as Schwinger functions, which, if satisfied, allowed one to completely reconstruct a quantum field theory in the sense of Wightman.

The Schwinger Functions

The Schwinger functions, S_n, are assumed to possess the following properties:

1. Distributions:

$$S_0 = 1 \qquad \text{and} \qquad S_n \in {}^0\mathscr{S}'(\mathbb{R}^{4n})$$

2. Euclidean Invariance:

$$S_n(f) = S_n(f_{(a,R)}) = S_n(f(Rx - a))$$

for all $R \in SO_4$, $a \in \mathbb{R}^4$, and $f \in {}^0\mathscr{S}(\mathbb{R}^{4n})$.

3. Positivity:

$$\sum_{n,m} S_{n+m}(\Theta f_n^* \times f_m) \geq 0$$

for all $\mathbf{f} \in \mathscr{S}_+$.

The OS Reconstruction Theorem

Using the distribution nature of the S_n in conjunction with the positivity condition allows one to define a positive semi-definite form

$$(\mathbf{f}, \mathbf{g}) = \sum_{n,m} S_{n+m}(\Theta f_n^* \times g_m)$$

on $\mathscr{S}_+ \times \mathscr{S}_+$.

The relativistic quantum field theory Hilbert space is obtained by taking the completion after modding out by all zero norm vectors.

Scattering Theory

A primary question in scattering theory pertains to the existence of scattering states: for any free state $\mathbf{f}_{0,-}$, does there exist a "scattering state" $\mathbf{f}_-$ that looks like the free state in the distant past.

This is formulated mathematically as

$$\lim_{t \to -\infty} \| U(t, I)\mathbf{f}_- - J_{\mathcal{A}} U_{\mathcal{A}}(t, I)\mathbf{f}_{0,-} \| = 0$$

The Cook Condition

A brief calculation provides a sufficient condition for the existence of scattering states and is known as Cook's condition:

$$\int_{-\infty}^{0} \|(HJ_{\mathcal{A}} - J_{\mathcal{A}}H_{\mathcal{A}})e^{iH_{\mathcal{A}}t}\mathbf{f}_{0,-}\| \, dt < \infty$$

In the case of two field scattering in the Euclidean representation, it was found that Cook's condition can be verified directly.

Establishing this result rests on the Källén-Lehmann spectral representation of the two-point Schwinger function S_2.

Weierstrass' Theorem

Completeness of Polynomials on Compact Intervals

Let $f \in L^2_\alpha([a,b])$, where $[a,b]$ is a *finite* interval. For every $\epsilon > 0$, there is a polynomial $p(x)$ such that

$$\int_a^b |f(x) - p(x)|^2 d\alpha(x) < \epsilon$$

Key idea is to use compactness of the interval to get uniform convergence.

All Is Not Well, Though

Example

Consider

$$w(x) = e^{-x^{\mu}\cos(\mu\pi)}dx \qquad \text{and} \qquad f(x) = \sin(x^{\mu}\sin(\mu\pi)),$$

where $0 < \mu < \frac{1}{2}$. Then $0 \neq f \in L^2_w([0,\infty))$. However,

$$\int_0^{\infty} f(x)x^n w(x)dx = 0 \qquad n \geq 0$$

from which it follows that f cannot be approximated to arbitrary accuracy using polynomials.

"The discussion of this [orthogonality] condition is closely connected with the uniqueness of Stieltjes' problem of moments." (Szego, Page 40)

The Moment Problem

The Stieltjes Problem

Given a sequence $\{m_n\}$ of "moments," does there exist a *unique* measure $\mu(x)$ on the half-line $[0, \infty)$ such that

$$m_n = \int_0^\infty x^n d\mu(x) \qquad \forall n \geq 0$$

Existence Condition (Akhiezer, Page 76)

The Stieltjes moment problem has a solution if and only if the two forms

$$\sum_{i,k=0}^{n} m_{i+k} x_i x_k \qquad \& \qquad \sum_{i,k=0}^{n} m_{i+k+1} x_i x_k$$

are non-negative for any n.

The Infamous Carleman Condition

Carleman's Condition

For the Stieltjes moment problem, a sufficient condition for determinacy is

$$\sum_{n=1}^{\infty} m_n^{-\frac{1}{2n}} = \infty$$

"Carleman has shown in [2] that this is true."