Fourier and Harmonic Analysis of Measures

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Goal

Given a Borel measure μ on $\mathbb{R}$, understand

$$\widehat{L^2(\mu)} = \{\widehat{f} : f \in L^2(\mu)\}.$$

Classic examples:

1. Lebesgue measure on $\mathbb{R}$:

$$\widehat{L^2(\mathbb{R})} = L^2(\mathbb{R}), \quad \text{(Plancherel);}$$

2. Lebesgue measure on $[-1/2, 1/2]$:

$$\widehat{L^2(-\frac{1}{2}, \frac{1}{2})} = PW_\pi, \quad \text{(Paley-Wiener);}$$

3. (Not-so classic) purely discrete: $\mu = \sum_n \delta_{x_n}$

$$\widehat{L^2(\mu)} = \{\text{AP-functions with frequencies in } \{x_n\}\} \text{ (Besicovitch).}$$

Two-Weight Inequality

For a fixed measure μ, determine the measures ν for which

$$\mathcal{F} : L^2(\mu) \to L^2(\nu)$$

1. is bounded,
2. is an isometry,
3. is unitary.

Jorgensen and Pedersen consider (3) specifically, especially ν discrete–μ is a spectral probability measure:

$$\|\hat{f}\|_\nu^2 = \sum_n |\hat{f}(x_n)|^2 = \sum_n |\langle f(t), e^{2\pi i t x_n}\rangle|^2 = \|f\|_\mu^2.$$

Bochner's Theorem

When is $F \in \widehat{L^2(\mu)}$?

1. Q: When is a function F the Fourier transform of something?
   1. Note: we are not placing any restriction on the "something".
   2. A: Bochner-Schoenberg-Eberlein conditions.
2. Q: When is a function F the Fourier transform of something in $L^2(\mu)$?
   1. Note: here we are *a priori* fixing μ.
   2. A: Open.

Other Ideas

1. Decay rates of $\hat{f}$
   1. Erdös et. al.
   2. "Strichartz Estimates"

$$\int_{\mathbb{R}^d} |f(x)|^2 d\mu \simeq \limsup_{R\to\infty} \frac{1}{R^{d-\alpha}} \int_{B_0(R)} |\widehat{f\, d\mu}(t)|^2 d\lambda.$$

2. Balayage
   1. Beurling
   2. Benedetto
3. Spectral Synthesis
4. Fourier Series in $L^2(\mu)$
   1. "Mock" Fourier of Strichartz
   2. Pseudo-continuable functions (Poltoratskii, Herr-W)
5. Fourier inversion (sampling theory of Strichartz)

Two-Weight Inequalities

Definition

We say that a Borel measure ν is a *Bessel measure* for μ if there exists a constant $B > 0$ such that for every $f \in L^2(\mu)$, we have

$$\|\widehat{f\, d\mu}\|^2_{L^2(\nu)} \leq B \|f\|^2_{L^2(\mu)}.$$

We say the measure ν is a *frame measure* for μ if there exists constants $A, B > 0$ such that for every $f \in L^2(\mu)$, we have

$$A\|f\|^2_{L^2(\mu)} \leq \|\widehat{f\, d\mu}\|^2_{L^2(\nu)} \leq B \|f\|^2_{L^2(\mu)}.$$

$\mathcal{F}$ is an isometry if $A = B = 1$.

Fourier Frames

Definition

For a finite Borel measure μ, a *Fourier frame* is a sequence $\{\omega_n e^{2\pi i x_n t}\}_n \subset L^2(\mu)$ such that there exists A, B satisfying:

$$A\|f\|_\mu^2 \leq \sum_n |\langle f, \omega_n e_{x_n} \rangle_\mu|^2 \leq B\|f\|_\mu^2.$$

If μ has a Fourier frame, then the measure

$$\nu = \sum_n |\omega_n| \delta_{x_n}$$

is a frame measure for μ, and

$$\mathcal{F} : L^2(\mu) \to L^2(\nu)$$

is bounded with a Moore-Penrose inverse.

Paley-Wiener space

For Lebesgue measure on $[-1/2, 1/2]$:

1. Duffin and Schaeffer (1952)
2. equivalent to the sampling problem for PW_π
    1. Shannon-Whitaker-Kotelnikov ($\sim$ 1940)
    2. Beurling density, Landau inequalities
3. also equivalent to the renormalization problem in PW_π
4. solved completely by Ortega-Cerdá and Seip (2002)

Cantor Measures

The middle-thirds Cantor set C_3 and invariant measure μ_3 generated by:

$$\phi_0(x) = \frac{x}{3} \quad \phi_1(x) = \frac{x+2}{3}$$

Cantor set C_4 and invariant measure μ_4 generated by:

$$\psi_0(x) = \frac{x}{4} \quad \psi_1(x) = \frac{x+2}{4}$$

Jorgensen and Pedersen (1998):

1. μ_4 is spectral,
2. spectrum is $\{0, 1, 4, 5, 16, 17, 20, 21, \dots\}$:
3. representation of Cuntz algebra $\mathcal{O}_2$, spectral theory of Ruelle operators,
4. μ_3 is not spectral.

Big open problem: Does μ_3 have a Fourier frame?

Beurling Dimension of Frame Measures

Frame Measures

Theorem (Dutkay, Han, & W.)

There exist finite compactly supported Borel measures that do not admit frame measures.

Theorem (Dutkay, Han, & W.)

If a measure μ has a Bessel/frame measure ν then it has also an atomic one.

Theorem (Dutkay, Han, & W.)

If ν is a frame measure for μ and $r > 0$ is sufficiently small, then $\{c_k e_{x_k} : k \in \mathbb{Z}^d\}$ is a weighted Fourier frame for μ, where $x_k \in r(k + Q)$ and $c_k = \sqrt{\nu(r(k + Q))}$.

Beurling Dimension

Definition

Let Q be the unit cube $Q = [0,1)^d$. For a locally finite measure ν and $\alpha \geq 0$ we define the *α-upper Beurling density* by

$$\mathcal{D}_\alpha(\nu) := \limsup_{R\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\nu(x+RQ)}{R^\alpha}.$$

We define the *(upper) Beurling dimension* of ν by

$$\dim_{\mathcal{B}} \nu := \sup\{\alpha \geq 0 : \mathcal{D}_\alpha(\nu) = \infty\}.$$

Beurling Dimension

Theorem (Dutkay, Han, Sun, & W.)

Let μ be a occasionally-α-dimensional measure and suppose ν is a Bessel measure for μ. Then $\mathcal{D}_\alpha(\nu) < \infty$ and so $\dim_{\mathcal{B}} \nu \leq \alpha$.

Definition

We say that a Borel measure μ is *ocasionally-α-dimensional* if there exists a sequence of Borel subsets E_n and some constants $c_1, c_2 > 0$ such that $diam(E_n)$ decreases to 0 as $n \to \infty$,

$$\sup_n \frac{diam(E_n)}{diam(E_{n+1})} < \infty$$

$$c_1 diam(E_n)^\alpha \leq \mu(E_n) \leq c_2 diam(E_n)^\alpha, \quad (n \geq 0).$$

Beurling Dimension

Shortcomings of Beurling Dimension:

1. not a complete description
2. upper bound necessary, no lower bound necessary
3. no sufficiency conditions are possible:
   the dimension measures geometric concentration of the measure, but not the precise location of large densities
4. No answer for μ_3.

Fourier Frames for μ_4:

Dilation of the Cantor-4 Set

We define a "dilated" iterated function system

$$\Upsilon_0(x,y) = (\frac{x}{4}, \frac{y}{2}), \qquad \Upsilon_1(x,y) = (\frac{x+2}{4}, \frac{y}{2})$$
$$\Upsilon_2(x,y) = (\frac{x}{4}, \frac{y+1}{2}), \qquad \Upsilon_3(x,y) = (\frac{x+2}{4}, \frac{y+1}{2}).$$

The corresponding invariant set is $C_4 \times [0,1]$ with invariant measure $\mu_4 \times \lambda$.

Filters

We choose filters

$$M_0(x, y) = H_0(x, y)$$
$$M_1(x, y) = e^{2\pi ix} H_1(x, y)$$
$$M_2(x, y) = e^{4\pi ix} H_2(x, y)$$
$$M_3(x, y) = e^{6\pi ix} H_3(x, y)$$

where

$$H_j(x, y) = \sum_{k=0}^{3} a_{jk} \chi_{\Upsilon_k(C_4 \times [0,1])}(x, y).$$

Filters (cont'd)

We require the following matrix to be unitary:

$$\mathcal{M}(x,y) = \frac{1}{2}\begin{pmatrix} M_0(\Upsilon_0(x,y)) & M_0(\Upsilon_1(x,y)) & M_0(\Upsilon_2(x,y)) & M_0(\Upsilon_3(x,y)) \\ M_1(\Upsilon_0(x,y)) & M_1(\Upsilon_1(x,y)) & M_1(\Upsilon_2(x,y)) & M_1(\Upsilon_3(x,y)) \\ M_2(\Upsilon_0(x,y)) & M_2(\Upsilon_1(x,y)) & M_2(\Upsilon_2(x,y)) & M_2(\Upsilon_3(x,y)) \\ M_3(\Upsilon_0(x,y)) & M_3(\Upsilon_1(x,y)) & M_3(\Upsilon_2(x,y)) & M_3(\Upsilon_3(x,y)) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ e^{\pi ix/4}a_{10} & -e^{\pi ix/4}a_{11} & e^{\pi ix/4}a_{12} & -e^{\pi ix/4}a_{03} \\ e^{\pi ix}a_{20} & e^{\pi ix}a_{21} & e^{\pi ix}a_{22} & e^{\pi ix}a_{23} \\ e^{3\pi ix/2}a_{30} & -e^{3\pi ix/2}a_{31} & e^{3\pi ix/2}a_{32} & -e^{3\pi ix/2}a_{33} \end{pmatrix}.$$

Filters (cont'd)

Factoring out the exponentials, we obtain two matrices:

$$\mathcal{M} = \frac{1}{2}\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & -a_{11} & a_{12} & -a_{03} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & -a_{31} & a_{32} & -a_{33} \end{pmatrix} \qquad \mathcal{H} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{03} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$$

unitary $\qquad\qquad$ $\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}^T$ in kernel

We also require $a_{00} = a_{01} = a_{02} = a_{03} = 1$.

Dilated Cuntz Isometries

We define isometries on $L^2(\mu_4 \times \lambda)$ for $j = 0, 1, 2, 3$ as follows:

$$\begin{aligned} S_j f(x, y) &= M_j(x, y) f(R(x, y)) \\ &= e^{2\pi i x j} H(x, y) f(R(x, y)). \end{aligned}$$

where $R(x, y) = (4x \mod 1, 2y \mod 1)$.
Since $\mathcal{M}$ is unitary, these isometries satisfy the Cuntz relations. Then, we can define an orthonormal set for $L^2(\mu_4 \times \lambda)$ by:

$$\{S_{\underline{j}} \mathbb{1} : \underline{j} \text{ is a reduced word in the alphabet } \{0, 1, 2, 3\}\}.$$

Fourier Frames

Theorem (Picioroaga & W. (2016))

For $|\rho| = 1$, $\rho \neq -1$

$$\left\{\omega_n e^{2\pi i n x} : n \in \mathbb{N}_0\right\}$$

is a Parseval frame in $L^2(\mu_4)$, *where* $\omega_n = \left(\frac{1+\rho}{2}\right)^{l_1(n)} 0^{l_2(n)} \left(\frac{1-\rho}{2}\right)^{l_3(n)}$.

Here, $l_k : \mathbb{N}_0 \to \mathbb{N}_0$ by $l_k(n)$ is the number of digits equal to k in the base 4 expansion of n. Note that $l_k(0) = 0$, and we follow the convention that $0^0 = 1$.

$$\mathcal{H} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & \rho & \rho \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -\rho & -\rho \end{pmatrix} \quad \mathcal{M}_\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \rho & -\rho \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\rho & \rho \end{pmatrix}$$

Projection

We define the subspace of $L^2(\mu_4 \times \lambda)$

$$V = \{f : \exists g \in L^2(\mu_4) \text{ with } f(x,y) = g(x)\}.$$

We then identify V with $L^2(\mu_4)$ in the canonical way.
We have:

$$\{P_V S_{\underline{j}} \mathbb{1}\}$$

is a Parseval frame for V.

$$P_V S_{\underline{j}} \mathbb{1} = \left(\frac{1+\rho}{2}\right)^{l_1(n)} 0^{l_2(n)} \left(\frac{1-\rho}{2}\right)^{l_3(n)} e^{2\pi i n x}$$

Here, $n = \sum_{i=1}^{K} j_i 4^{K-i}$ when $\underline{j} = j_K \dots j_1$.

Projection (cont'd)

Lemma

If $f(x,y) = g(x)h(x,y)$ *with* $g \in L^2(\mu_4)$ *and* $h \in L^\infty(\mu_4 \times \lambda)$*, then*

$$[P_V f](x,y) = g(x)H(x)$$

where $H(x) = \int_{[0,1]} h(x,y) d\lambda(y)$*.*

Lemma

For any reduced word $\omega = j_K j_{K-1} \dots j_1$*,*

$$\int \prod_{k=1}^{K} H_{j_k}(R^{k-1}(x,y)) \, d\lambda(y) = \prod_{k=1}^{K} \int H_{j_k}(4^{k-1}x, y) \, d\lambda(y).$$

What about the Cantor-3 Set?

Answer: It doesn't work.

Impossible to choose coefficients $\mathcal{H}$ to obtain:

1. $\mathcal{M}$ unitary
2. $\begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}^T$ in the kernel
3. the first row identically 1!

Fourier Series without Frames

Kacmarz Algorithm

Given $\{\varphi_n\}_{n=0}^{\infty} \subset H$ and $\langle x, \varphi_n \rangle$, can we recover x? Note: yes if ONB/frame.

$$
\begin{aligned}
x_0 &= \langle x, \varphi_0 \rangle \varphi_0 \\
x_n &= x_{n-1} + \langle x - x_{n-1}, \varphi_n \rangle \varphi_n .
\end{aligned}
$$

If $\lim_{n\to\infty} \|x - x_n\| = 0$ for all x, then the sequence $\{\varphi_n\}_{n=0}^{\infty}$ is said to be effective.

$$x = \sum \langle x, g_i \rangle \varphi_i .$$

Fourier Series

Theorem (Herr & W., 2015)

If μ is a singular Borel probability measure on $[0, 1)$, then the sequence $\left\{e^{2\pi inx}\right\}_{n=0}^{\infty}$ is effective in $L^2(\mu)$. As a consequence, any element $f \in L^2(\mu)$ possesses a Fourier series

$$f(x) = \sum_{n=0}^{\infty} c_n e^{2\pi inx},$$

where the sum converges in the $L^2(\mu)$ norm.

$$c_n = \int_0^1 f(x)\overline{g_n(x)}\, d\mu(x),$$

where $\{g_n\}_{n=0}^{\infty}$ is the auxiliary sequence of $\left\{e^{2\pi inx}\right\}_{n=0}^{\infty}$ in $L^2(\mu)$.

Inversion Lemma

Lemma (Herr & W., 2015)

There exists a sequence $\{\alpha_n\}_{n=0}^{\infty}$ *such that*

$$g_n(x) = \sum_{j=0}^{n} \overline{\alpha_{n-j}} e^{2\pi i j x}.$$

For each $n \in \mathbb{N}$, *let* $P_n = \left\{(m_1, \ldots, m_k) \in \mathbb{N}^k \;\middle|\; k \in \mathbb{N},\; m_1 + m_2 + \cdots + m_k = n\right\}$. *For* $p \in P_n$, *let* $\ell(p)$ *be the length of* p. *Let* μ *be a Borel probability measure on* $[0,1)$ *with Fourier-Stieltjes transform* $\widehat{\mu}(x) = \int_{[0,1)} e^{-2\pi i x y}\, d\mu(y)$. *Define* $\alpha_0 = 1$, *and for* $n \geq 1$, *let*

$$\alpha_n = \sum_{p \in P_n} (-1)^{\ell(p)} \prod_{j=1}^{\ell(p)} \widehat{\mu}(p_j).$$

Elucidation of Fourier Coefficients

Corollary

Let μ be a singular Borel probability measure on $[0,1)$, let $\{g_n\}$ be the auxiliary sequence of $\{e^{2\pi inx}\}$ in $L^2(\mu)$, and with respect to these, let $\{\alpha_n\}_{n=0}^{\infty}$ be the sequence of scalars from the Inversion Lemma. Then for any $f \in L^2(\mu)$,

$$f(x) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \alpha_{n-j} \widehat{f}(j) \right) e^{2\pi inx},$$

where the convergence is in norm, and

$$\widehat{f}(j) := \int_0^1 f(x) e^{-2\pi ijx} \, d\mu(x).$$

Harmonic Analysis of Measures:
Reproducing Kernels

The Hardy Space

The classical Hardy space $H^2(\mathbb{D})$ consists of those holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ satisfying

$$\|f\|_{H^2}^2 := \sup_{0<r<1} \int_0^1 \left| f(re^{2\pi i x}) \right|^2 \, dx < \infty.$$

Equivalently,

$$H^2 = \left\{ \sum_{n=0}^{\infty} c_n z^n \;\middle|\; \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\},$$

with norm

$$\left\| \sum_{n=0}^{\infty} c_n z^n \right\|_{H^2}^2 = \sum_{n=0}^{\infty} |c_n|^2 .$$

Boundary Functions

Definition

Let ν be a finite Borel measure on $[0,1)$, and let $F : \mathbb{D} \to \mathbb{C}$ be a member of the Hardy space H^2. For each $0 < r < 1$, define $F_r : [0,1) \to \mathbb{C} \in L^2(\nu)$ by

$$F_r(x) := F(re^{2\pi i x}).$$

We say that F possesses an $L^2(\nu)$ **boundary function** $F^\star$ if there exists a function $F^\star : [0,1) \to \mathbb{C} \in L^2(\nu)$ such that

$$\lim_{r \to 1^-} \|F_r - F^\star\|_\nu = 0.$$

Let λ denote Lebesgue measure on $[0,1)$. It is known that every member F of H^2 possesses an $L^2(\lambda)$ boundary. Moreover,

$$\langle F, G \rangle_{H^2} = \langle F^\star, G^\star \rangle_\lambda .$$

The Szegő Kernel

The Hardy space is a reproducing kernel Hilbert space (RKHS). Its kernel is the Szegő kernel

$$s_z(w) := \frac{1}{1 - \overline{z}w}.$$

Thus for all $F \in H^2$,

$$F(z) = \langle F, s_z \rangle_{H^2} = \langle F^\star, s_z^\star \rangle_\lambda .$$

In particular,

$$s_z(w) = \langle s_z^\star, s_w^\star \rangle_\lambda .$$

Thus the Hardy space's kernel reproduces with respect to $L^2(\lambda)$ boundaries. Indeed, the kernel of any closed subspace will also do so.

Searching for a Non-Spectral Analogue

Dutkay and Jorgensen (2011) demonstrate that for spectral measures μ, there exists a subspace of the Hardy space whose kernel reproduces itself with respect to μ.

Q: Do there exist kernels that reproduce themselves with respect to $L^2(\mu)$ boundaries if μ is non-spectral?

Lemma (Jorgensen & W.)

If $\left\{e^{2\pi i \gamma x}\right\}_{\gamma \in \Gamma \subseteq \mathbb{N}_0}$ *is a Fourier frame in* $L^2(\mu)$*, then*

$$K_z(w) := \sum_{\gamma \in \Gamma} \sum_{\gamma' \in \Gamma} \langle d_\gamma, d_{\gamma'} \rangle \, \overline{z}^{\gamma} w^{\gamma'}$$

is such a kernel, where $\{d_\gamma\}_{\gamma \in \Gamma}$ *is a dual frame of* $\{e_\gamma\}_{\gamma \in \Gamma}$.

How about μ_3?

"Big" Open Questions

Definition ($\mathcal{K}(\mu)$)

Given a Borel measure μ on $[0,1)$, we define $\mathcal{K}(\mu)$ to be the set of positive matrices K on $\mathbb{D}$ such that

$$K_z(w) := \int_0^1 K_z^\star(x) \overline{K_w^\star(x)} \, d\mu(x)$$

for all $z, w \in \mathbb{D}$.

Definition ($\mathcal{M}(K)$)

Given a positive matrix K on $\mathbb{D}$, we define $\mathcal{M}(K)$ to be the set of nonnegative Borel measures μ on $[0,1)$ such that for each fixed $z \in \mathbb{D}$, K_z possesses an $L^2(\mu)$ boundary $K_z^\star$ and $K_z(w)$ reproduces itself with respect to integration of these $L^2(\mu)$ boundaries.

Q1: Which $K \subset H^2$ are in $\mathcal{K}(\mu)$? Q2: Which μ are in $\mathcal{M}(K)$ if $K \subset H^2$?

Herglotz Representation Theorem and the space $\mathcal{H}(b)$

Theorem

There is a 1-to-1 correspondence between the nonconstant inner functions b in H^2 and the nonnegative singular Borel measures μ on $\mathbb{T} \equiv [0,1)$ given by

$$Re\left(\frac{1+b(z)}{1-b(z)}\right) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2}\, d\mu(\xi).$$

We will say that b corresponds to μ, and that μ corresponds to b. The construction of the de Branges-Rovnyak space $\mathcal{H}(b)$ is based on Toeplitz operators, but here suffice it to say that for b an inner function, we have

$$\mathcal{H}(b) = H^2 \ominus bH^2.$$

$\mathcal{H}(b)$ as a μ-RKHS

Theorem (Herr & W., 2015)

Let μ be a singular Borel probability measure with corresponding inner function b, and let k_z^b the kernel of $\mathcal{H}(b)$. Then

$$k_z^b(w) = \frac{1 - \overline{b(z)}b(w)}{1 - \overline{z}w} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle g_n, g_m \rangle_{\mu} \overline{z}^n w^m.$$

Every function $F \in \mathcal{H}(b)$ is then given by

$$F(z) = \int_0^1 F^\star(x) \overline{(k_z^b)^\star(x)} \, d\mu(x),$$

where $F^\star$ denotes the $L^2(\mu)$ boundary of the function F.

Implications

This means that for any nonnegative singular Borel measure μ with corresponding inner function b,

$$k^b \in \mathcal{K}(\mu) \text{ and } \mu \in \mathcal{M}(k^b).$$

Let V be a subspace of $\mathcal{H}(b)$ and P_V the orthogonal projection onto it. Then

$$P_V k_z^b \in \mathcal{K}(\mu) \text{ and } \mu \in \mathcal{M}(P_V k_z^b).$$

However, there are more examples than these!

Dextroduals and Levoduals

Definition

Given a Hilbert space $\mathbb{H}$ and two sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ in $\mathbb{H}$, if we have

$$\sum_{n=0}^{\infty} \langle f, x_n \rangle \, y_n = f$$

with convergence in norm for all $f \in \mathbb{H}$, then $\{x_n\}_{n=0}^{\infty}$ is said to be dextrodual to $\{y_n\}_{n=0}^{\infty}$ (or, "a dextrodual of $\{y_n\}_{n=0}^{\infty}$"), and $\{y_n\}_{n=0}^{\infty}$ is said to be levodual to $\{x_n\}_{n=0}^{\infty}$.

$\{x_n\}$ and $\{y_n\}$ needn't be frames. Indeed, $\{g_n\}$ is dextrodual to $\{e_n\}$ in $L^2(\mu)$ for any singular measure μ on $[0, 1)$, but $\{e_n\}$ is not even Bessel.

Reproducing Kernels from Dextroduals

Theorem (Herr, Jorgensen, & W., 2015)

Let μ be a Borel measure on $[0,1)$. Let $\{\psi_n\} \subset L^2(\mu)$ be a Bessel sequence that is dextrodual to $\{e_n\}$. Then for each fixed $z \in \mathbb{D}$,

$$K_z(w) := \sum_m \sum_n \langle \psi_n, \psi_m \rangle_\mu \overline{z}^n w^m$$

is a well-defined function on $\mathbb{D}$. Consequently, $K_z \in \mathcal{K}(\mu)$.

Some Dextroduals of $\{e_n\}$

There exist many dextroduals of $\{e_n\}$ in $L^2(\mu)$:

Theorem (Herr, Jorgensen, & W., 2015)

Suppose μ and λ are singular Borel probability measures on $[0,1)$ such that $\mu << \lambda$, and suppose there exist constants A and B such that $0 < A \leq \frac{d\mu}{d\lambda} \leq B$ on supp $\left(\frac{d\mu}{d\lambda}\right) := \left\{x \in [0,1) \mid \frac{d\mu}{d\lambda}(x) \neq 0\right\}$. If $\{h_n\}$ is the auxiliary sequence of $\{e_n\}_{n=0}^{\infty}$ in $L^2(\lambda)$, then for all $f \in L^2(\mu)$,

$$f = \sum_{n=0}^{\infty} \left\langle f, \frac{h_n}{\frac{d\mu}{d\lambda}} \right\rangle_{\mu} e_n$$

in the $L^2(\mu)$ norm. Moreover, $\left\{\frac{h_n}{\frac{d\mu}{d\lambda}}\right\}$ is a frame in $L^2(\mu)$ with bounds no worse than $\frac{1}{B}$ and $\frac{1}{A}$. Furthermore, if λ' also satisfies the hypotheses, then $\lambda' \neq \lambda$ implies $\left\{\frac{h_n'}{\frac{d\mu}{d\lambda'}}\right\} \neq \left\{\frac{h_n}{\frac{d\mu}{d\lambda}}\right\}$ in $L^2(\mu)$.

Sub-Hardy spaces

Consequence: There exist subspaces of $H^2(\mathbb{D})$–with a different norm!–such that the kernel reproduces itself with respect to μ.

In the Dutkay-Jorgensen spectral situation, the norms are equal.

The End

Thank you!