Comparing Two Generalized Nevanlinna-Pick Theorems

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Outline

Classical Nevanlinna-Pick Theorem

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.
Let $H^\infty(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \mid f$ is bounded and analytic$\}$.

Theorem (Pick 1915)

Given N distinct points $z_1, \ldots, z_N \in \mathbb{D}$ and N points $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, there exists $f \in H^\infty(\mathbb{D})$ such that $||f|| \leq 1$ and

$$f(z_i) = \lambda_i,\ i = 1, \ldots, N,$$

if and only if the Pick matrix

$$\left[\frac{1-\overline{\lambda_i}\lambda_j}{1-\overline{z_i}z_j}\right]_{i,j=1}^N$$

is positive semidefinite.

Classical Nevanlinna-Pick Theorem

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Definitions

W^*-algebra

A W^*-**algebra** M is a C^*-algebra that is a dual space.

W^*-correspondence

A W^*-**correspondence** E over a W^*-algebra M is a self-dual right Hilbert C^*-module over M equipped with a normal $*$-homomorphism $\varphi : M \to \mathscr{L}(E)$ that gives the left action of M on E.

Examples of W^*-correspondences

- $M = E = \mathbb{C}$
  - $a \cdot c \cdot b = acb$
  - $\langle c, d \rangle = \overline{c}d$
- $M = \mathbb{C},\ E = \mathbb{C}^N$
  - $a \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix} \cdot b = \begin{bmatrix} ac_1 b \\ \vdots \\ ac_N b \end{bmatrix}$
  - $\left\langle \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix}, \begin{bmatrix} d_1 \\ \vdots \\ d_N \end{bmatrix} \right\rangle = \sum \overline{c_i} d_i$

Examples Cont.

- $G = (G^0, G^1, r, s)$, $M = C(G^0)$, $E = C(G^1)$
  - $(a \cdot \xi \cdot b)(e) = a(r(e))\xi(e)b(s(e))$
  - $\langle \xi, \eta \rangle (v) = \sum_{s(e)=v} \overline{\xi(e)}\eta(e)$

W^*-Correspondence Setting

Given

- M, a W*-algebra
- E, a W*-correspondence over M

define

- the **Fock space** $\mathscr{F}(E)$ to be the direct sum $\bigoplus_{k=0}^{\infty} E^{\otimes k}$, where $E^{\otimes 0} = M$, viewed as a bimodule over itself
- the von Neumann algebra of **bounded operators** $\mathscr{L}(\mathscr{F}(E))$ on the Fock space of E

Operators on the Fock Space $\mathscr{F}(E)$

Define the **left action operator** $\varphi_\infty : M \to \mathscr{L}(\mathscr{F}(E))$ by

$$\varphi_\infty(a) = \begin{bmatrix} a & & & \\ & \varphi(a) & & \\ & & \varphi^{(2)}(a) & \\ & & & \ddots \end{bmatrix}$$

where $\varphi^{(k)}(a) : E^{\otimes k} \to E^{\otimes k}$ is given by

$$\varphi^{(k)}(a)(\xi_1 \otimes \xi_2 \otimes \ldots \xi_k) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \ldots \xi_k.$$

Operators on the Fock Space $\mathscr{F}(E)$ Cont.

For $\xi \in E$, define the **left creation operator** $T_\xi : \mathscr{F}(E) \to \mathscr{F}(E)$ by $T_\xi(\eta) = \xi \otimes \eta$. As a matrix,

$$T_\xi = \begin{bmatrix} 0 & & & \\ T_\xi^{(1)} & 0 & & \\ & T_\xi^{(2)} & 0 & \\ & & \ddots & \ddots \end{bmatrix}$$

where $T_\xi^{(k)} : E^{\otimes k-1} \to E^{\otimes k}$ is given by

$$T_\xi^{(k)}(\eta_1 \otimes \ldots \otimes \eta_{k-1}) = \xi \otimes \eta_1 \otimes \ldots \otimes \eta_{k-1}.$$

Subalgebras of $\mathscr{L}(\mathscr{F}(E))$

Tensor Algebra of E

The **tensor algebra** of E, denoted $\mathscr{T}_+(E)$, is defined to be the norm-closed subalgebra of $\mathscr{L}(\mathscr{F}(E))$ generated by $\{\varphi_\infty(a) \mid a \in M\}$ and $\{T_\xi \mid \xi \in E\}$.

Hardy Algebra of E

The **Hardy algebra** of E, denoted $H^\infty(E)$, is defined to be the ultraweak closure of $\mathscr{T}_+(E)$ in $\mathscr{L}(\mathscr{F}(E))$.

The σ-dual E^σ

Given

- (M, E)
- $\sigma : M \to B(H)$, a faithful, normal representation of M on a Hilbert space H,

define

- the **induced representation** $\sigma^E : \mathscr{L}(E) \to B(E \otimes_\sigma H)$ by $\sigma^E(T) = T \otimes I_H$
- σ**-dual** $E^\sigma := \{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = \sigma^E \circ \varphi(a)\eta \text{ for all } a \in M\}$

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E^σ Cont.

$$E^\sigma := \{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = \sigma^E \circ \varphi(a)\eta \text{ for all } a \in M\}$$

E^σ is a W^*-correspondence over $\sigma(M)'$. For $a, b \in \sigma(M)'$ and $\eta, \xi \in E^\sigma$, define

- $a \cdot \eta \cdot b := (I_E \otimes a)\eta b$
- $\langle \eta, \xi \rangle := \eta^* \xi$

We define $H^\infty(E^\sigma)$ analogously.

E^σ Cont.

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Algebra of Upper Triangular Toeplitz Operators

$\mathscr{U}_{\mathscr{T}}(E, H, \sigma)$

Define $\mathscr{U}_{\mathscr{T}}(E, H, \sigma)$ to be the algebra of upper triangular "Toeplitz" operators $T \in \mathscr{L}(\mathscr{F}(E) \otimes_\sigma H)$ such that

- $$T = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} & \cdots \\ 0 & I_E \otimes T_{00} & I_E \otimes T_{01} & I_E \otimes T_{02} & \cdots \\ 0 & 0 & I_{E^{\otimes 2}} \otimes T_{00} & I_{E^{\otimes 2}} \otimes T_{01} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$
- $T_{0j}(\varphi^{(j)}(a) \otimes I_H) = \sigma(a) T_{0j}$ for all $a \in M$.

$\mathscr{U}_{\mathscr{T}}(E, H, \sigma)^* = \rho(H^\infty(E^\sigma))$

Define $U : \mathscr{F}(E^\sigma) \otimes_\iota H \to \mathscr{F}(E) \otimes_\sigma H$ by

$$U(\eta_1 \otimes \cdots \otimes \eta_k \otimes h) = (I_{E^{\otimes k-1}} \otimes \eta_1) \cdots (I_E \otimes \eta_{k-1})\eta_k h.$$

Define $\rho : H^\infty(E^\sigma) \to B(\mathscr{F}(E) \otimes_\sigma H)$ by

$$\rho(X) = U(X \otimes I_H)U^*.$$

Then

$$\mathscr{U}_{\mathscr{T}}(E, H, \sigma)^* = \rho(H^\infty(E^\sigma)).$$

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Define $\rho : H^\infty(E^\sigma) \to B(\mathscr{F}(E) \otimes_\sigma H)$ by

$$\rho(X) = U(X \otimes I_H)U^*.$$

Then

$$\mathscr{U}_{\mathscr{T}}(E, H, \sigma)^* = \{X \otimes I_H \mid X \in H^\infty(E)\}' = \rho(H^\infty(E^\sigma)),$$

where the second equality is due to Muhly and Solel [4, Theorem 3.9].

Cauchy Kernel

$$E^\sigma := \{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = \sigma^E \circ \varphi(a)\eta \text{ for all } a \in M\}$$

For $\eta \in E^\sigma$ and $k \in \mathbb{N}$, define

- $\eta^{(k)} \in B(H, E^{\otimes k} \otimes_\sigma H)$ by

$$\eta^{(k)} = (I_{E^{\otimes k-1}} \otimes \eta)(I_{E^{\otimes k-2}} \otimes \eta) \cdots (I_E \otimes \eta)\eta$$

- the **Cauchy Kernel** $C(\eta) \in B(H, \mathscr{F}(E) \otimes_\sigma H)$ by

$$C(\eta) = \begin{bmatrix} I_H & \eta & \eta^{(2)} & \eta^{(3)} & \dots \end{bmatrix}^T$$

Point Evaluation

Point Evaluation

For $X \in H^\infty(E^\sigma)$ and $\eta \in E^\sigma$ with $\|\eta\| < 1$, define the **point evaluation** $\hat{X}(\eta) \in \sigma(M)'$ by

$$\begin{aligned} \hat{X}(\eta) &= \langle \rho(X) C(0), C(\eta) \rangle \\ &= C(0)^* \rho(X)^* C(\eta), \end{aligned}$$

where $C(0) = \begin{bmatrix} I_H & 0 & 0 & \cdots \end{bmatrix}^T$ is the Cauchy kernel at 0.

Observations about Point Evaluation

$$\hat{X}(\eta) = \langle \rho(X)C(0), C(\eta) \rangle,$$

- Not multiplicative, ie if $X, Y \in H^\infty(E^\sigma)$ and $\eta \in E^\sigma$ with $\|\eta\| < 1$, then
$$\widehat{XY}(\eta) \neq \hat{X}(\eta)\hat{Y}(\eta)$$
- Induces an algebra antihomomorphism from $H^\infty(E^\sigma)$ into the completely bounded maps on $\sigma(M)'$

Observations Cont.

Definition

For $X \in H^\infty(E^\sigma)$ and $\eta \in E^\sigma$ with $\|\eta\| < 1$, define Φ_X^η on $\sigma(M)'$ by

$$\Phi_X^\eta(a) := \langle C(\eta), \rho(\varphi_\infty^\sigma(a))\rho(X)C(0)\rangle, \quad a \in \sigma(M)'.$$

- Φ_X^η is a completely bounded map on $\sigma(M)'$.
- The map $X \mapsto \Phi_X^\eta$ is an algebra antihomomorphism on $H^\infty(E^\sigma)$, ie $\Phi_{XY}^\eta = \Phi_Y^\eta \circ \Phi_X^\eta$.

Generalized Nevanlinna-Pick Theorem

Theorem (N. 2016)

For $i = 1, \ldots, N$, let $\mathfrak{z}_i \in E^\sigma$ with $\|\mathfrak{z}_i\| < 1$ and $\Lambda_i \in \sigma(M)'$. There exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that

$$\hat{X}(\mathfrak{z}_i) = \Lambda_i,\ i = 1, \ldots, N,$$

if and only if the operator matrix

$$\left[C(\mathfrak{z}_i)^*(I_{\mathscr{F}(E)} \otimes (I_H - \Lambda_i^*\Lambda_j))C(\mathfrak{z}_j)\right]_{i,j=1}^N$$

is positive semidefinite.

Corollary I

Let $M = \mathbb{C}, E = \mathbb{C}^d$, and $\sigma : M \to B(H)$ be given by $\sigma(a) = aI_H$. Then $E^\sigma = R_d(B(H))$ and $\sigma(M)' = B(H)$.

Theorem (Constantinescu and Johnson 2003)

For $i = 1, \ldots, N$, let $\mathfrak{z}_i \in R_d(B(H))$ with $\|\mathfrak{z}_i\| < 1$ and $\Lambda_i \in B(H)$. There exists $X \in H^\infty(R_d(B(H)))$ with $\|X\| \leq 1$ such that

$$\hat{X}(\mathfrak{z}_i) = \Lambda_i, \; i = 1, \ldots, N,$$

if and only if the operator matrix

$$\left[C(\mathfrak{z}_i)^*(I_{\mathscr{F}(E)} \otimes (I_H - \Lambda_i^* \Lambda_j)) C(\mathfrak{z}_j) \right]_{i,j=1}^N$$

is positive semidefinite.

Corollary II

Let $M = E = \mathbb{C}$ and $\sigma : M \to B(\mathbb{C})$ be given by $\sigma(a) = a$.
Then $E^\sigma = \mathbb{C}$ and $\sigma(M)' = \mathbb{C}$.

Theorem (Pick 1915)

For $i = 1, \ldots, N$, let $\mathfrak{z}_i \in \mathbb{D}$ and $\Lambda_i \in \mathbb{C}$. There exists $X \in H^\infty(\mathbb{C})$ with $\|X\| \leq 1$ such that

$$\hat{X}(\mathfrak{z}_i) = \Lambda_i,\ i = 1, \ldots, N,$$

if and only if the matrix

$$\left[\frac{1-\overline{\Lambda_i}\Lambda_j}{1-\overline{\mathfrak{z}_i}\mathfrak{z}_j}\right]_{i,j=1}^{N}$$

is positive semidefinite.

Displacement Equation

Given $\Phi : B(H) \to B(H)$ with $\|\Phi\| < 1$ and $B \in B(H)$, define the **displacement equation**

$$(I_{B(H)} - \Phi)(A) = B.$$

Since $\|\Phi\| < 1$, solve for A by computing the Neumann series

$$\begin{aligned} A &= (I_{B(H)} - \Phi)^{-1}(B) \\ &= \sum_{k=0}^{\infty} \Phi^k(B). \end{aligned}$$

We are interested in the case when Φ is completely positive. In this case, $(I_{B(H)} - \Phi)^{-1}$ is completely positive as well.

Proof of Generalized NP Theorem (N. 2015)

Step 1

Let $\mathfrak{z} = \begin{bmatrix} \mathfrak{z}_1 & & \\ & \ddots & \\ & & \mathfrak{z}_N \end{bmatrix}$, $U = \begin{bmatrix} I_H \\ \vdots \\ I_H \end{bmatrix}$, and $V = \begin{bmatrix} \Lambda_1^* \\ \vdots \\ \Lambda_N^* \end{bmatrix}$, and form the displacement equation

$$(I_{B(H)} - \Phi_{\mathfrak{z}})(A) = UU^* - VV^*,$$

where $\Phi_{\mathfrak{z}}(A) = \mathfrak{z}^*(I_E \otimes A)\mathfrak{z}$, and $\|\Phi_{\mathfrak{z}}\| < 1$ because $\|\mathfrak{z}\| < 1$.

Proof Cont.

Step 2

Observe

- The Pick matrix

$$A = \left[C(\mathfrak{z}_i)^* (I_{\mathscr{F}(E)} \otimes (I_H - \Lambda_i^* \Lambda_j)) C(\mathfrak{z}_j) \right]_{i,j=1}^N$$

  is the unique solution of the displacement equation.
- We can rewrite the Pick matrix as $A = U_\infty^* U_\infty - V_\infty^* V_\infty$, where $U_\infty = \begin{bmatrix} C(\mathfrak{z}_1) & \cdots & C(\mathfrak{z}_N) \end{bmatrix}$ and $V_\infty = \begin{bmatrix} (I_{\mathscr{F}(E)} \otimes \Lambda_1) C(\mathfrak{z}_1) & \cdots & (I_{\mathscr{F}(E)} \otimes \Lambda_N) C(\mathfrak{z}_N) \end{bmatrix}$.

Proof Cont.

Lemma (Step 3)

$A = U_\infty^* U_\infty - V_\infty^* V_\infty$ is positive if and only if there exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that $\rho(X)^* U_\infty = V_\infty$.

Step 4

$\rho(X)^* U_\infty = V_\infty$ if and only if $\hat{X}(\mathfrak{z}_i) = \Lambda_i$ for all i.

Proof of Lemma

Lemma

$A = U_\infty^* U_\infty - V_\infty^* V_\infty$ is positive if and only if there exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that $\rho(X)^* U_\infty = V_\infty$.

Proof: $(\Rightarrow)$

- $A \geq 0 \Rightarrow \exists L \in \sigma^{(N)}(M)'$ such that $A = LL^*$
- Displacement equation becomes

$$\begin{bmatrix} L & V \end{bmatrix} \begin{bmatrix} L^* \\ V^* \end{bmatrix} = \begin{bmatrix} \mathfrak{z}^*(I_E \otimes L) & U \end{bmatrix} \begin{bmatrix} (I_E \otimes L^*)\mathfrak{z} \\ U^* \end{bmatrix}$$

$$\hat{A}^* \hat{A} = \hat{B}^* \hat{B}$$

Proof of Lemma Cont.

- Douglas's Lemma $\Rightarrow \exists!$ partial isometry $\theta = \begin{bmatrix} X & Z \\ Y & W \end{bmatrix}$ such that
  - $\hat{A} = \theta \hat{B}$
  - $Inn(\theta) \subseteq \overline{Range(\hat{B})}$
  - $\|\theta\| = 1$
- Define

$$T = \begin{bmatrix} W & Y(I_E \otimes Z) & Y(I_E \otimes X)(I_{E^{\otimes 2}} \otimes Z) & \cdots \\ 0 & I_E \otimes W & I_E \otimes Y(I_E \otimes Z) & \cdots \\ 0 & 0 & I_{E^{\otimes 2}} \otimes W & \cdots \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

Proof of Lemma Cont.

Then

- $T \in \mathscr{U}_{\mathscr{T}}(E, H, \sigma)$
- $\|T\| \leq 1$ because T is the transfer map of the contractive time-varying system

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = (I_{E^{\otimes t}} \otimes \theta) \begin{bmatrix} x(t+1) \\ u(t) \end{bmatrix}$$

- $TU_\infty = V_\infty$

Thus there exists $X \in H^\infty(E^\sigma)$ with $\|X\| \leq 1$ such that $T = \rho(X)^*$, and $\rho(X)^* U_\infty = V_\infty$.

Proof of Lemma Cont.

($\Leftarrow$) If there exists $X \in H^\infty(E^\sigma)$ such that $\|X\| \leq 1$ and $\rho(X)^* U_\infty = V_\infty$, then

$$\begin{aligned} A &= U_\infty^* U_\infty - V_\infty^* V_\infty \\ &= U_\infty^* U_\infty - U_\infty^* \rho(X)\rho(X)^* U_\infty \\ &= U_\infty^*(I - \rho(X)\rho(X)^*)U_\infty \\ &\geq 0 \end{aligned}$$

since $\|X\| \leq 1$ and ρ is an isometry.

M-S Generalized Nevanlinna-Pick Theorem

Theorem (Muhly and Solel 2004)

For $i = 1, \ldots, N$, let $\mathfrak{z}_i \in E^\sigma$ with $\|\mathfrak{z}_i\| < 1$ and $\Lambda_i \in B(H)$. There exists $Y \in H^\infty(E)$ with $\|Y\| \leq 1$ such that

$$\hat{Y}(\mathfrak{z}_i^*) = \Lambda_i, \quad i = 1, \ldots, N,$$

if and only if the map from $M_N(\sigma(M)')$ to $M_N(B(H))$ defined by

$$\Phi = \left[(I_{B(H)} - Ad(\Lambda_i, \Lambda_j)) \circ (I_{B(H)} - \Phi_{\mathfrak{z}_i, \mathfrak{z}_j})^{-1}\right]_{i,j=1}^N$$

is completely positive.

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is completely positive.

Comparing the Generalized Nevanlinna-Pick Theorems

Theorem (N. 2016)

For $i = 1, \ldots, N$, let $\mathfrak{z}_i \in \mathfrak{Z}(E^\sigma)$ with $\|\mathfrak{z}_i\| < 1$ and $\Lambda_i \in \mathfrak{Z}(\sigma(M)')$. The following are equivalent:

1. There exists $Y \in H^\infty(\mathfrak{Z}(E))$ with $\|Y\| \leq 1$ such that
$$\hat{Y}(\mathfrak{z}_i^*) = \Lambda_i^*, \quad i = 1, \ldots, N$$
in the sense of Muhly and Solel's theorem.
2. There exists $X \in H^\infty(\mathfrak{Z}(E^\sigma))$ with $\|X\| \leq 1$ such that
$$\hat{X}(\mathfrak{z}_i) = \Lambda_i, \quad i = 1, \ldots, N$$
in the sense of Norton's theorem.

Definition

Center of a W^*-correspondence

If E is a W^*-correspondence over a W^*-algebra M, then the **center** of E, denoted $\mathfrak{Z}(E)$, is the collection of points $\xi \in E$ such that $a \cdot \xi = \xi \cdot a$ for all $a \in M$, and $\mathfrak{Z}(E)$ is a W^*-correspondence over $\mathfrak{Z}(M)$.

What's next?

- Compare Popescu's Generalized Nevanlinna-Pick Theorem to Constantinescu and Johnson's
- Study Popescu's proof
- Use the displacement theorem to prove a commutant lifting theorem?

Popescu's setting

Let $S_1, \ldots, S_n$ be the left creation operators on the Fock Space of $\mathbb{C}^n$. For $\Phi \in H^\infty(\mathbb{C}^n) \otimes B(H)$, we can write

$$\Phi = \sum_{\alpha \in F_n^+} S_\alpha \otimes A_\alpha, \quad A_\alpha \in B(H).$$

Popescu's Point Evaluation

Let $\mathfrak{Z} = (Z_1, \ldots, Z_n)$, where $Z_i \in B(H)$ and $\sum Z_i Z_i^* < rI_H$, $0 < r < 1$. Define the **point evaluation** of Φ at $\mathfrak{Z}$ by

$$\Phi(\mathfrak{Z}) = \sum_{\alpha \in F_n^+} Z_\alpha A_\alpha.$$

Popescu's Generalized Nevanlinna-Pick Theorem

Theorem (Popescu 2003)

For $j = 1, \ldots, m$, let $B_j, C_j \in B(H)$ and

$$\mathfrak{Z}_j = [Z_{j1}, \ldots, Z_{jn}] : H^{\oplus n} \to H, \text{ with } r(\mathfrak{Z}) < 1.$$

There exists $\Phi \in H^\infty(\mathbb{C}^n) \otimes B(H)$ such that $\|\Phi\| \leq 1$ and

$$[(I_{\mathscr{F}(\mathbb{C}^n)} \otimes B_j)\Phi](\mathfrak{Z}_j) = C_j, \quad j = 1, \ldots, m$$

if and only if the operator matrix

$$P_P = \left[\sum_{k=0}^{\infty} \sum_{|\alpha|=k} Z_{j\alpha}[B_j B_k^* - C_j C_k^*](Z_{k\alpha})^*\right]_{j,k=1}^{m}$$

is positive semidefinite.

Proof via Constantinescu and Johnson's theorem

Step 1

Define $\tilde{\mathfrak{z}}_j = [Z_{j1}^*, \dots, Z_{jn}^*]$. By Constantinescu and Johnson's theorem, there exists $T \in \mathscr{U}_{\mathscr{T}}(\mathbb{C}^n, H, \sigma)$ such that $B_j T(\tilde{\mathfrak{z}}_j) = C_j^*$ if and only if

$$P_{CJ} = \left[C(\tilde{\mathfrak{z}}_j)^* (I_{\mathscr{F}(E)} \otimes (B_j^* B_k - C_j^* C_k)) C(\tilde{\mathfrak{z}}_k) \right]_{j,k=1}^{m}$$

is positive semidefinite.

Step 2

The Pick matrices are equal.

Proof Cont.

Step 3

There exists $T \in \mathscr{U}_{\mathscr{T}}(\mathbb{C}^n, H, \sigma)$ with $\|T\| \leq 1$ and such that $B_j T(\tilde{\mathfrak{z}}_j) = C_j^* \iff$ there exists $\Phi \in H^\infty(\mathbb{C}^n) \otimes B(H)$ has $\|\Phi\| \leq 1$ and satisfies $[(I_{\mathscr{F}(\mathbb{C}^n)} \otimes B_j)\Phi](\mathfrak{z}_j) = C_j$.
Hint: $\Phi := (J \otimes I_H) T^* (J \otimes I_H)$.

Flipping Isomorphism

Define the "flipping" isomorphism $J : \mathscr{F}(\mathbb{C}^n) \to \mathscr{F}(\mathbb{C}^n)$ by $J(e_{i_1} \otimes \cdots \otimes e_{i_k}) = e_{i_k} \otimes \cdots \otimes e_{i_1}$.

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