Uniqueness theorems for combinatorial C^*-algebras

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Let $\mathscr{G}$ be a graph, k-graph, or groupoid, and $C^*(\mathscr{G})$ the universal C*-algebra defined from it.

Question: Under what circumstances is a $*$-homomorphism $\phi : C^*(\mathscr{G}) \to B(H)$ injective?

Classical theorems addressing this question assume either

(a) the existence of intertwining “gauge actions" on the algebras (*Gauge Invariant Uniqueness Theorem* [1]), or

(b) an aperiodicity condition on the graph itself (*Cuntz-Krieger Uniqueness Theorem* [2]),

and conclude that ϕ is injective iff it is *nondegenerate*, i.e., injective on the “diagonal subalgebra” $\mathcal{D}$.

[1] an-Huef, ‘97

[2] Fowler-Kumjian-Pask-Raeburn, ‘97

Theorem (Brown-Nagy-R-Sims-Williams) There is a canonical subalgebra $\mathscr{M} \subset C^*(\mathscr{G})$ such that a $*$-homomorphism $\phi : C^*(\mathscr{G}) \to B(H)$ is injective iff $\phi|_{\mathscr{M}}$ is injective.

Moreover, $\mathscr{M} \subset C^*(\mathscr{G})$ is a Cartan inclusion.

[NR1] Nagy and Reznikoff, *Abelian core of graph algebras*, J. Lond. Math. Soc. (2) **85** (2012), no. 3, 889–908.

[NR2] Nagy and Reznikoff, *Pseudo-diagonals and uniqueness theorems*, Proc. AMS (2013).

[BNR] Brown, Nagy, Reznikoff *A generalized Cuntz-Krieger uniqueness theorem for higher-rank graphs*, JFA (2013).

[BNRSW] Brown, Nagy, Reznikoff, Sims, and Williams, *Cartan subalgebras of groupoid C^*-algebras* (2015).

Drinen (1999): Every AF algebra is Morita equivalent to a graph algebra.

Kumjian-Pask-Raeburn (1998)

- The algebra is AF iff the graph has no cycles.
- All simple graph algebras are AF or purely infinite.

Hong-Szymański (2004): the ideal structure of the algebra can be completely described from the graph.

Generalizations and related constructions: Exel crossed product algebras, Leavitt path algebras (Abrams, Ruiz, Tomforde), topological graph algebras (Katsura), Ruelle algebras (Putnam, Spielberg), Exel-Laca algebras, ultragraphs (Tomforde), Cuntz-Pimsner algebras, higher-rank Cuntz-Krieger algebras (Robertson-Steger), etc.

k-graph algebras (Kumjian and Pask, 2000)

• developed to generalize graph algebras and higher-rank Cuntz-Krieger algebras,
• whether simple, purely infinite, or AF can be determined from properties of the graph (Kumjian-Pask, Evans-Sims),
• are groupoid C*-algebras,
• include examples of algebras that are simple but neither AF nor purely infinite, and hence not graph algebras (Pask-Raeburn-Rordam-Sims),
• include examples that can be constructed from shift spaces (Pask-Raeburn-Weaver),
• can be used to construct any Kirchberg algebra (Spielberg).

Let $k \in \mathbb{N}^+$. We regard $\mathbb{N}^k$ as a category with a single object, 0, and with composition of morphisms given by addition.

A ***k*-graph** is a countable category Λ along with a "degree" functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *unique factorization property*:

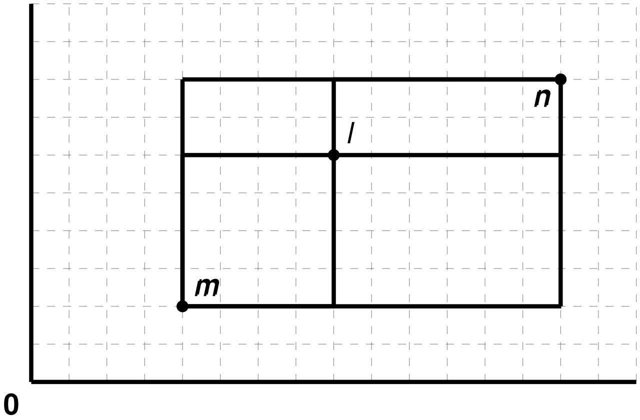
> For all $\lambda \in \Lambda$, and $m, n \in \mathbb{N}^k$, if $d(\lambda) = m + n$ then there are unique $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$.

- Denote the range and source maps $r, s : \Lambda \to \Lambda$.
- Refer to objects as *vertices* and morphisms as *paths*.
- We assume: for all $v \in \Lambda, \;\; n \in \mathbb{N}^k$, $0 < |r^{-1}(\{v\}) \cap d^{-1}(\{n\})| < \infty$.

Example The set of finite paths in a directed graph, with $d(\alpha) =$ the length of α, forms a 1-graph.

Example: Rectangles in $\mathbb{N}^k$

Let $\Omega_k := \{(l, n) \in \mathbb{N}^k \times \mathbb{N}^k \mid l \leq n\}$ with $d(l, n) = n - l$, $s(m, l) = l = r(l, n)$, and $(m, l)(l, n) = (m, n)$.



A **Cuntz-Krieger Λ-family** in a C*-algebra A is a set $\{T_\lambda, \lambda \in \Lambda\}$ of partial isometries in A satisfying

(i) $\{T_v \mid v \in d^{-1}(\{0\})$ is a family of mutually orthogonal projections,

(ii) $T_{\lambda\mu} = T_\lambda T_\mu$ for all λ, $\mu \in \Lambda$ s.t. $s(\lambda) = r(\mu)$,

(iii) $T_\lambda^* T_\lambda = T_{s(\lambda)}$ for all $\lambda \in \Lambda$, and

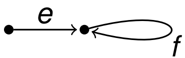
(iv) for all $v \in d^{-1}(\{0\})$ and $n \in \mathbb{N}^k$, $\quad T_v = \sum_{\substack{d(\lambda)=n \\ r(\lambda)=v}} T_\lambda T_\lambda^*$.

$C^*(\Lambda)$ will denote the C*-algebra generated by a universal Cuntz-Krieger Λ-family, $(S_\lambda, \lambda \in \Lambda)$, with $P_\lambda = S_\lambda S_\lambda^*$.

Note: $C^*(\Lambda) = \overline{\text{span}}\{S_\alpha S_\beta^* \mid \alpha, \beta \in \Lambda \mid s(\alpha) = s(\beta)\}$
Defn: The diagonal $\mathcal{D} := \overline{\text{span}}\{S_\alpha S_\alpha^* \mid \alpha \in \Lambda\}$.

Classic uniqueness theorems

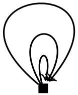
Coburn's Theorem ('67)



Any nondegenerate representation $C^*(T_e, T_f)$ is isomorphic to the Toeplitz algebra $\mathcal{T}$, generated by one non-unitary isometry.

Cuntz ('77) Any nondegenerate representation is isomorphic to the Cuntz Algebra $\mathcal{O}_n$, generated by n partial isometries S_i satisfying $\forall i,\ S_i^* S_i = \sum_{j=1}^{n} S_j S_j^*$.

n loops



Cuntz-Krieger ('80) Graph with 0-1 adjacency matrix A

When A satisfies a "fullness" condition (I), any nondegenerate representation is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_A$.

Is non-degeneracy enough?

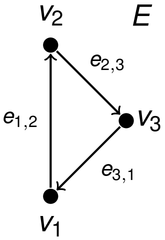
No! Consider the cycle of length three, E.
The map $\phi : C^*(E) \to M_3(\mathbb{C})$ given by

$$S_{v_i} \mapsto \varepsilon_{i,i} \qquad S_{e_{i,j}} \mapsto \varepsilon_{j,i}.$$

is a $*$-homomorphism. However

$$\phi(S_{e_{3,1}} S_{e_{2,3}} S_{e_{1,2}}) = \varepsilon_{1,3}\varepsilon_{3,2}\varepsilon_{2,1} = \varepsilon_{1,1} = \phi(S_{v_1}),$$

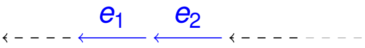
whereas $S_{e_{3,1}} S_{e_{2,3}} S_{e_{1,2}} \neq S_{v_1}$ in $C^*(E)$, so ϕ is not injective.



The infinite path space Λ^∞

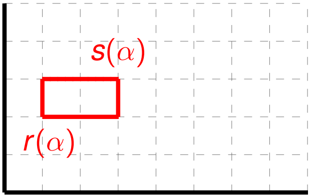
Defn. An infinite path in a k-graph Λ is a degree-preserving covariant functor $x : \Omega_k \to \Lambda$.

$k = 1$ picture



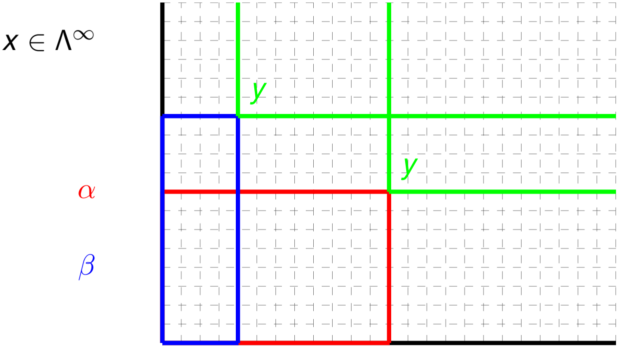
$x(1,3) = e_1 e_2$
$d(e_1 e_2) = 2$

$k = 2$ picture



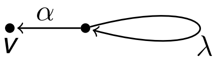
$x((1,2),(3,3)) = \alpha$
$d(\alpha) = (2,1)$

An infinite path x in a k-graph is *eventually periodic* if there are $\alpha \neq \beta$ in Λ and $y \in \Lambda^\infty$ such that $x = \alpha y = \beta y$; otherwise x is aperiodic.



Λ is **aperiodic** if every vertex is the range vertex of an aperiodic infinite path.

• In a directed graph, cycles without entry reveal failure of aperiodicity.



Clearly the only infinite path with range v, $\alpha\lambda\lambda\lambda\cdots$, is eventually periodic.

• A k-graph Λ is aperiodic iff $\mathcal{D}$ is a masa.

Cuntz-Krieger Uniqueness Theorem:
(Kumjian-Pask-Raeburn-Fowler, et. al. ('90's))
When ϕ is nondegenerate and the graph satisfies

(L) every cycle has an entry

then ϕ is injective.

Theorem Szymański (2001), Nagy-R (2010):
Condition (L) can be replaced with a condition on the spectrum of $\phi(S_\lambda)$ for cycles λ without entry.

• Uniqueness theorems of Raeburn-Sims-Yeend and Kumjian-Pask assume aperiodicity of the k-graph.

Theorem Nagy-Brown-R (2013, JFA)
A $*$-homomorphism $\phi : C^*(\Lambda) \to \mathcal{A}$ is injective iff it is injective on the subalgebra $\mathscr{M} := C^*(S_\alpha S_\beta^* \,|\, \forall \gamma \in \Lambda^\infty \;\alpha\gamma = \beta\gamma\}$.

Theorem Yang (2014)

- $\mathscr{M} = \mathcal{D}'$.
- For fixed $m, n \in \mathbb{N}^k$, and $\alpha_0 \in \Lambda$ with $d(\alpha) = n$, and an element $X = \sum S_\alpha S_\beta^*$ of $\mathscr{M}$, there is at most one term $S_{\alpha_0} S_{\beta_0}^*$ with $d(\beta_0) = m$.

A **groupoid** $\mathcal{G}$ is a small category in which every element has an inverse. When $\mathcal{G}$ is a topological groupoid, $C^*(\mathcal{G})$ is defined to be a completion of $C_c(\mathcal{G})$.

The isotropy subgroupoid is the set
$\mathrm{Iso}(\mathcal{G}) := \{g \in \mathcal{G} \mid r(g) = s(g)\}$

Theorem (Brown-Nagy-R-Sims-Williams, 2014)
Let $\mathcal{G}$ be a locally compact, amenable, Hausdorff, étale groupoid, with $(\mathrm{Iso}\,\mathcal{G})^\circ$ is closed. If $\phi : C^*(\mathcal{G}) \to A$ is a C^*-homomorphism, then the following are equivalent.

(i) ϕ is injective.
(ii) ϕ is injective on $\mathcal{M} := C^*((\mathrm{Iso}(\mathcal{G}))^\circ)$.

Assume G is a 2nd countable locally compact Hausdorff étale groupoid. For $f \in C_c(G)$, define

$$f^*(\gamma) = \overline{f(\gamma^{-1})} \qquad f * g\,(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta).$$

Define the I-norm on $C_c(G)$ by

$$\|f\|_I = \sup_{u \in G^{(0)}} \max\left\{ \sum_{\gamma\in G_u} |f(\gamma)|, \sum_{\gamma\in G^u} |f(\gamma)| \right\}, \text{ and let}$$

$$\|f\| = \sup\{\|\pi(f)\| \mid \pi \text{ is an } I\text{-norm bounded } *\text{-rep. of } C_c(G)\}$$

$\boldsymbol{C^*(G)}$ is the completion of $C_c(G)$ with respect to this norm.

Given a k-graph Λ, define

$$\mathcal{G}_\Lambda = \{(\alpha y, d, \beta y) \mid y \in \Lambda^\infty, \alpha, \beta \in \Lambda, s_\Lambda(\alpha) = s_\Lambda(\beta) = r_\Lambda(y), \\ d = d_\Lambda(\beta) - d_\Lambda(\alpha)\}$$

with

$$r(x, d, y) = x, \qquad s(x, d, y) = y$$
$$(x, d, y)(z, d', w) = \delta_y(z)(x, d + d', w).$$

• The cylinder sets $Z(\alpha, \beta) = \{(\alpha y, d, \beta z) \in \mathcal{G}_\Lambda\}$ form a basis for an étale groupoid.

• Moreover, $C^*(\Lambda) = C^*(\mathcal{G}_\Lambda)$ and $\mathcal{M} = C^*((\mathrm{Iso}(\mathcal{G}))^\circ)$ via the map $S_\alpha S_\beta^* \mapsto \chi_{Z(\alpha,\beta)}$.

(Renault, '80) A masa C*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is **Cartan** if

(i) $\exists$ a faithful conditional expectation $\mathcal{A} \to \mathcal{B}$,
(ii) The normalizer of $\mathcal{B}$ in $\mathcal{A}$ generates $\mathcal{A}$, and
(iii) $\mathcal{B}$ contains an approximate unit of $\mathcal{A}$.

Extension properties for pure states on masa $\mathcal{B} \subset \mathcal{A}$:
(UEP) Every pure state extends uniquely to $\mathcal{A}$.

(**AEP**) Densely many pure states extend uniquely.

Kadison-Singer Problem (Marcus-Spielman-Srivastava)
(UEP) holds for all masa in $B(\ell^2(\mathbb{N}))$.

Thm (Nagy-R, 2011; NRBSW, 2014) $\mathcal{M} \subseteq C^*(\mathcal{G})$ is Cartan.

Thm (NRBSW, 2014) All Cartan subalgebras satisfy the AEP.

Thank you!

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