Toward a Complete Pick Property in a W^*-Setting

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Classical Muhly/Solel

Definition

M, a W^*-algebra
E is a W^*-**correspondence over** M if

- E is a Hilbert W^*-module over M
- E is a left M-module via a unital W^*-homomorphism $\varphi : M \to \mathscr{L}(E)$, i.e.

$$a \cdot x = \varphi(a)(x) \qquad \forall a \in A, x \in E$$

The Fock Space

Tensor Powers

- $E^{\otimes 0} = M$
- $E^{\otimes 1} = E$
- $E^{\otimes k} = E \otimes E^{\otimes k-1}$ (Defined inductively for $k \geq 2$).
- $\mathscr{F}(E) = \sum_{k=0}^{\infty} \oplus\, E^{\otimes k}$ (The Fock Space)

New Correspondences

- $E^{\otimes k}$, left action: φ_k
- $\mathscr{F}(E)$, left action: φ_∞

Important elements of $\mathscr{L}(\mathscr{F}(E))$

Left action maps, $a \in M$

$$\varphi_\infty(a) = \begin{bmatrix} \varphi_0(a) & 0 & \cdots \\ 0 & \varphi_1(a) & \\ \vdots & & \ddots \end{bmatrix}_{i,j=0}^{\infty}$$

Creation operators, $\xi \in E^{\otimes k}$, $k \in \mathbb{N}$

$T_\xi(z) = \xi \otimes z$

$$T_\xi = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ T_\xi^{(0)} & 0 & \cdots & 0 \\ 0 & T_\xi^{(1)} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \end{bmatrix}_{i,j=0}^{\infty} \qquad (\text{Case } k = 1, \xi \in E)$$

Important Operator Algebras

Subalgebras of $\mathscr{L}(\mathscr{F}(E))$

- $\mathcal{T}_{0+}(E)$ ("algebraic tensor algebra")

  algebra generated by $\varphi_\infty(a)$, T_ξ
- $\mathcal{T}_{+}(E)$: ("tensor algebra")

  norm-closure of $\mathcal{T}_{0+}(E)$
- $\mathcal{H}^\infty(E)$: ("Hardy algebra")

  ultraweak closure of $\mathcal{T}_{0+}(E)$

Generalization: Weights!

- M: W^*-algebra
- E: W^*-correspondence over M

The R-Sequence

Let $\{R_k\}_{k=0}^{\infty}$ be a family such that

- $R_k \in \varphi_k(M)^c$, $\forall k \in \mathbb{N}$
- R_k is positive and invertible $\forall k \in \mathbb{N}$.
- $R_0 = I$.
- $\sup_{k \in \mathbb{N}} \|R_k^{-1}(I_E \otimes R_{k-1})\| < \infty$
- $\limsup_{k \to \infty} \|R_k\|^{1/k} < \infty$

The Z-Sequence

Define the "sequence of weights", $\{Z_k\}_{k=0}^{\infty}$,

$$Z_k = \begin{cases} I_M, & \text{if } k = 0 \\ R_k^{-1}(I_E \otimes R_{k-1}) & \text{if } k > 0 \end{cases}$$

Put 'em Together...

Left action maps, $a \in M$

$$\varphi_\infty(a) = \begin{bmatrix} \varphi_0(a) & 0 & \\ 0 & \varphi_1(a) & \\ & & \ddots \end{bmatrix}_{i,j=0}^{\infty}$$

Weighted Creation Operators, $\xi \in E^{\otimes k}, k \in \mathbb{N}$

$$W_\xi = \begin{bmatrix} 0 & & \\ Z_1 T_\xi^{(0)} & 0 & \\ 0 & Z_2 T_\xi^{(1)} & \\ & & \ddots \end{bmatrix}_{i,j=0}^{\infty} \qquad k = 1, \xi \in E,$$

Weighted Operator Algebras

Subalgebras of $\mathscr{L}(\mathscr{F}(E))$

- $\mathcal{T}_{0+}(E,Z)$: ("weighted algebraic tensor algebra")

  algebra generated by $\varphi_\infty(a)$, W_ξ

- $\mathcal{T}_{+}(E,Z)$: ("weighted tensor algebra")

  norm-closure of $\mathcal{T}_{0+}(E,Z)$

- $\mathcal{H}^\infty(E,Z)$: ("weighted Hardy algebra")

  ultraweak closure of $\mathcal{T}_{0+}(E,Z)$

Fundamental Idea

To understand an algebra, view it as an algebra of *functions* on its space of representations.

Motivating Question:

Can we paramaterize the completely bounded ultraweakly continuous representations of $\mathcal{H}^\infty(E,Z)$ on Hilbert space?

Getting Those Representations

The Natural Choice

- Fix $\sigma : M \to \mathscr{B}(H)$ a normal unital $*$-hom.
- Take $\mathfrak{z} \in \mathcal{I}(\sigma^E \circ \varphi, \sigma)$.
- Fact: There is a representation, $\rho : \mathcal{T}_{0+}(E, Z) \to \mathscr{B}(H)$ such that
  - $\rho(\varphi_\infty(a)) = \sigma(a)$
  - $\rho(W_\xi)(h) = \mathfrak{z}(\xi \otimes h)$.

Question:

Can we extend ρ to $\mathcal{H}^\infty(E, Z)$?

Partial answer

Muhly, Solel: Matricial Function Theory and Weighted Shifts

- Suppose $\sum_{k=0}^{\infty} \mathfrak{z}^{(k)}(R_k^2 \otimes I_H)\mathfrak{z}^{(k)*}$ converges ultraweakly in $\mathscr{B}(H)$.
- Define $\mathscr{R}_{\mathfrak{z}} : \sigma(M)' \to \sigma(M)'$
$$\mathscr{R}_{\mathfrak{z}}(A) = \sum_{k=0}^{\infty} \mathfrak{z}^{(k)}(R_k^2 \otimes A)\mathfrak{z}^{(k)*}$$

Then

- $\mathscr{R}_{\mathfrak{z}}$ is a linear completely positive map.
- IF there exists a completely positive map, $\mathscr{X}_{\mathfrak{z}} : \sigma(M)' \to \sigma(M)'$ such that $\|\mathscr{X}_{\mathfrak{z}}\| \leq 1$, and
$$\mathscr{R}_{\mathfrak{z}} = (\iota - \mathscr{X}_{\mathfrak{z}})^{-1},$$
then ρ extends to an ultraweakly continuous representation of $\mathcal{H}^{\infty}(E, Z)$,
- In particular: we are very happy.

A New Sequence, X

The X-Sequence

$$X_k = \begin{cases} 0, & \text{if } k = 0 \\ \sum_{l=1}^{k} (-1)^{l+1} \left(\sum_{\alpha} \bigotimes_{i=1}^{l} R^2_{\alpha(i)} \right) & \text{if } k \geq 1 \end{cases}$$

(summing over $\alpha : \{1, \ldots, l\} \to \mathbb{N}$ such that $\sum_{i=1}^{l} \alpha(i) = k$)

Idea behind the Formula

$$\sum_{k=0}^{\infty} r_k^2 t^k = \frac{1}{1 - \sum_{k=1}^{\infty} x_k t^k}$$

Recall:

. . . IF there exists a completely positive map, $\mathscr{X}_{\mathfrak{z}} : \sigma(M)' \to \sigma(M)'$ such that $\|\mathscr{X}_{\mathfrak{z}}\| \leq 1$, and

$$\mathscr{R}_{\mathfrak{z}} = (\iota - \mathscr{X}_{\mathfrak{z}})^{-1},$$

then ρ extends to an ultraweakly continuous representation of $\mathcal{H}^{\infty}(E, Z)$,

Here Comes the Complete Pick Property

The X-map

Suppose that $\mathscr{X}_{\mathfrak{z}} : \sigma(M)' \to \sigma(M)'$

$$\mathscr{X}_{\mathfrak{z}}(A) = \sum_{k=0}^{\infty} \mathfrak{z}^{(k)}(X_k \otimes A)\mathfrak{z}^{(k)*}$$

is well-defined. Ignoring convergence issues, $\mathscr{R}_{\mathfrak{z}} = (\iota - \mathscr{X}_{\mathfrak{z}})^{-1}$!!

Questions:

- Is $\mathscr{X}_{\mathfrak{z}}$ well-defined?
- Is $\|\mathscr{X}_{\mathfrak{z}}\| \leq 1$?
- Is $\mathscr{X}_{\mathfrak{z}}$ completely positive? If all X_k are positive: yes!!

Answer in Case $M = E = \mathbb{C}$ (McCullough-Quiggin, Agler, McCarthy)

$X_k \geq 0, \quad \forall k \geq 1 \qquad \Longleftrightarrow \qquad$ the associated kernel has the "Complete Pick Property"

Crash Course in RKHS's

Definition

A **reproducing kernel Hilbert space**, RKHS, is a Hilbert space, H, of functions on some set, Ω such that evaluation at each point of Ω is a non-zero continuous linear functional on H.

Definition

The **reproducing kernel at** $w \in \Omega$: $k_w \in H$ such that

$$f(w) = \langle f, k_w \rangle \quad \forall f \in H$$

Definition

The **reproducing kernel** associated with H: $\mathscr{K} : \Omega \times \Omega \to \mathbb{C}$,

$$\mathscr{K}(w, z) = \langle k_z, k_w \rangle$$

Definition

$\phi : \Omega \to \mathbb{C}$ is a **multiplier** of H if its pointwise product with any element of H also belongs to H.

Classical Interpolation Problem

Important Example: the Hardy Space, $H^2(\mathbb{D})$

- **Kernel**: $\mathscr{K} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$, $\mathscr{K}(w, z) = \frac{1}{1 - w\overline{z}}$
- **Multipliers**: $H^\infty(\mathbb{D})$

Question:

Fix $k \in \mathbb{N}$; $\{\omega_i\}_{i=1}^k, \{\lambda_i\}_{i=1}^k \subset \mathbb{D}$.

When is there a function, ϕ, in $H^\infty(\mathbb{D})$ of norm at most 1 that **interpolates** this data; i.e. when $1 \le i \le k$,

$$\phi(\omega_i) = \lambda_i?$$

Answer: (Nevanlinna/Pick (1915)) Happy 100th Anniversary!

Such a ϕ exists iff

$$\left[\mathscr{K}(\omega_i, \omega_j)(1 - \lambda_i\overline{\lambda_j})\right]_{i,j=1}^k = \left[\frac{1 - \lambda_i\overline{\lambda_j}}{1 - \omega_i\overline{\omega_j}}\right]_{i,j=1}^k \ge 0$$

Generalized Operator-Theoretic Interpolation Problem

Question:

- H: RKHS on Ω;
- $\mathscr{K}$: reproducing kernel
- $k \in \mathbb{N}$; $\{\omega_i\}_{i=1}^k \subset \Omega$, $\{\lambda_i\}_{i=1}^k \subset \mathbb{D}$.

When is there a multiplier, ϕ, of H of norm at most 1 such that

$$\phi(\omega_i) = \lambda_i \quad \forall 1 \leq i \leq k?$$

Definition

H has the **Pick property** if

$$\left[\mathscr{K}(\omega_i, \omega_j)(1 - \lambda_i \overline{\lambda_j})\right]_{i,j=1}^k \geq 0$$

implies the existance of an interpolating multiplier of norm ≤ 1.

- Example: The Hardy Space, $H^2(\mathbb{D})$
- Non-Example: The Bergman space

The Complete Pick Property

$M_{s\times t}$ Pick Property

- $\mathscr{K}$, a kernel, on Ω
- $s, t \in \mathbb{N}$

$\mathscr{K}$ has the $M_{s\times t}$ **Pick Property** if whenever

- $k \in \mathbb{N}$, $\{\omega_i\}_{i=1}^k \subseteq \Omega$, $\{W_i\}_{i=1}^k \subseteq M_{s\times t}(\mathbb{C})$
- $[\mathscr{K}(\omega_i, \omega_j)(I - W_i W_j^*)]_{i,j=1}^k \geq 0 \qquad$ (in $M_k(M_s(\mathbb{C}))$)

there exists $\Phi \in \text{Mult}(H \otimes \mathbb{C}^t, H \otimes \mathbb{C}^s)$, such that

$$\Phi(\omega_i) = W_i \quad \forall 1 \leq i \leq k$$

Definition

$\mathscr{K}$ has the **Complete Pick Property** if it has the $M_{s\times t}$ property $\forall s, t \in \mathbb{N}$.

- Example: Hardy Space

What Does That Have to Do with the Price of Eggs?

Theorem (Agler, McCarthy)

An (irreducible) kernel, $\mathscr{K}$, is a Complete Pick kernel if and only if

$$\mathscr{K}(\zeta,\lambda) = \frac{\overline{\delta(\zeta)}\delta(\lambda)}{1 - F(\zeta,\lambda)}$$

for a positive semi-definite function $F : \Omega \times \Omega \to \mathbb{D}$ and a nowhere vanishing function $\delta : \Omega \to \mathbb{C}$

Recall:

... IF there exists a completely positive map, $\mathscr{X}_{\mathfrak{z}} : \sigma(M)' \to \sigma(M)'$ such that $\|\mathscr{X}_{\mathfrak{z}}\| \leq 1$, and

$$\mathscr{R}_{\mathfrak{z}} = (\iota - \mathscr{X}_{\mathfrak{z}})^{-1},$$

then ρ extends to an ultraweakly continuous representation of $\mathcal{H}^{\infty}(E, Z)$

Conclusion

In the one-dimensional case, we can use the Complete Pick Property.

The Goal

Formulate a W^*-Complete Pick Property

Generalizing the Classical Theory

Reproducing Kernel Hilbert Space (RKHS)

- H: a Hilbert space
- H is a subspace of $F(\Omega, \mathbb{C})$, for a set, Ω.
- (Positive Kernel) $K : \Omega \times \Omega \to \mathbb{C}$ such that

$$[\mathscr{K}(\omega_i, \omega_j)]_{ij=1}^n \geq 0$$

for any $n \in N$, $\{\omega_i\}_i^n \subseteq \Omega$.

Reproducing Kernel W^*-Correspondence (RKWC)

- E: an (M, N) W^*-correspondence, for W^*-algebras, M, N.
- E is a sub N-module of $F(M \times \Omega, N)$, for a set, Ω.
- (Normal Completely Positive Kernel) $\mathscr{K} : \Omega \times \Omega \to \mathscr{B}_{uw}(M, N)$ such that

$$[\mathscr{K}(\omega_i, \omega_j)(a_i a_j^*)]_{ij=1}^n \geq 0,$$

for any $n \in \mathbb{N}$, $\{\omega_i\}_{i=1}^n \subseteq \Omega$, $\{a_i\}_{i=1}^n \subseteq M$.

Generalizing the Classical Theory

Multipliers

- $\phi : \Omega \to \mathbb{C}$ is a **multiplier of** H if $\phi \cdot f \in H$, $\forall f \in H$,

$$\phi \cdot f(\omega) = \phi(\omega) f(\omega)$$

- $Mult(H)$: the multipliers of H

W^*-Multipliers

- $\phi : \Omega \to N$ is a **multiplier of** E if $\phi \cdot f \in E$, $\forall f \in E$,

$$\phi \cdot f(a, \omega) = \phi(\omega) f(a, \omega)$$

- $Mult(E)$: the multipliers of E.

Matrix-Valued Multipliers

$\Phi : \Omega \to M_{s \times t}(\mathbb{C})$ is an (s,t)-**multiplier** if $\exists \{\phi_{mn}\} \subset Mult(H)$, $\forall \omega \in \Omega$,

$$\Phi(\omega) = [\phi_{mn}(\omega)]_{m=1}^{s} {}_{n=1}^{t}$$

Matrix-Valued W^*-Multipliers (G)

$\Phi : \Omega \to M_{s \times t}(N)$ is an (s,t)-**multiplier** if $\exists \{\phi_{mn}\} \subset Mult(E)$, $\forall \omega \in \Omega$,

$$\Phi(\omega) = [\phi_{mn}(\omega)]_{m=1}^{s} {}_{n=1}^{t}$$

Generalizing the Classical Theory

Recall: $M_{s\times t}$ Pick Property, $s, t \in \mathbb{N}$

$\mathscr{K} : \Omega \times \Omega \to \mathbb{C}$, a positive kernel, has the $M_{s\times t}$ **Pick Property** if

- $k \in \mathbb{N}$, $\{\omega_i\}_{i=1}^k \subseteq \Omega$, $\{W_i\}_{i=1}^k \subseteq M_{s\times t}(\mathbb{C})$
- $[\mathscr{K}(\omega_i, \omega_j)(I - W_i W_j^*)]_{i,j=1}^k \geq 0 \qquad$ (in $M_k(M_s(\mathbb{C}))$)

implies there exists an (s,t) multiplier, Φ, of norm ≤ 1 such that

$$\Phi(\omega_i) = W_i \quad \forall 1 \leq i \leq k$$

Observation

$$[\mathscr{K}(\omega_i, \omega_j)(I - W_i W_j^*)]_{i,j=1}^k =$$

$$\left[\underbrace{\begin{bmatrix} \mathscr{K}(\omega_i,\omega_j) & & \\ & \ddots & \\ & & \mathscr{K}(\omega_i,\omega_j) \end{bmatrix}}_{(s\times s)} - W_i \underbrace{\begin{bmatrix} \mathscr{K}(\omega_i,\omega_j) & & \\ & \ddots & \\ & & \mathscr{K}(\omega_i,\omega_j) \end{bmatrix}}_{(t\times t)} W_j^* \right]_{i,j=1}^k$$

A W^*-formulation of the Complete Pick Property

W^*-$M_{s\times t}$ Pick Property, $s, t \in \mathbb{N}$ (G)

$\mathscr{K} : \Omega \times \Omega \to \mathscr{B}_{uw}(M, N)$, NCP kernel, has the $M_{s\times t}$ **Pick Property** if whenever

- $k \in \mathbb{N}$,
- $\{\omega_i\}_{i=1}^k \subseteq \Omega$,
- $\{B_i\}_{i=1}^k \subseteq M_{s\times s}(N)$,
- $\{D_i\}_{i=1}^k \subseteq M_{s\times t}(N)$
- $\mathscr{A} : M_k(M) \to M_k(M_s(N))$ is completely positive, where

$$\mathscr{A}\left([a_{ij}]_{ij}^k\right) =$$

$$\left[B_i \begin{bmatrix} \mathscr{K}(\mathfrak{w}_i, \mathfrak{w}_j)(a_{ij}) & & \\ & \ddots & \\ & & \mathscr{K}(\mathfrak{w}_i, \mathfrak{w}_j)(a_{ij}) \end{bmatrix} B_j^* - D_i \begin{bmatrix} \mathscr{K}(\mathfrak{w}_i, \mathfrak{w}_j)(a_{ij}) & & \\ & \ddots & \\ & & \mathscr{K}(\mathfrak{w}_i, \mathfrak{w}_j)(a_{ij}) \end{bmatrix} D_j^* \right]_{ij}^k$$

There exists an (s, t) multiplier, Φ, of norm ≤ 1 such that

$$B_i \Phi(\omega_i) = D_i, \quad 1 \leq i \leq k.$$

A W^*-formulation of the Complete Pick Property

Complete Pick NCP Kernel

$\mathscr{K} : \Omega \times \Omega \to \mathscr{B}_{uw}(M, N)$, an NCP kernel, is a **Complete Pick NCP Kernel** if it has the $M_{s \times t}$ Pick Property $\forall s, t \in \mathbb{N}$

Question:

Are there any examples?

Finding a Complete Pick NCP Kernel

Recall:

… IF there exists a completely positive map, $\mathscr{X}_{\mathfrak{z}} : \sigma(M)' \to \sigma(M)'$ such that $\|\mathscr{X}_{\mathfrak{z}}\| \leq 1$, and

$$\mathscr{R}_{\mathfrak{z}} = (\iota - \mathscr{X}_{\mathfrak{z}})^{-1},$$

then ρ extends to an ultraweakly continuous representation of $\mathcal{H}^{\infty}(E, Z)$

Can't win the game? Change the Rules

- Our approach: $R \to X$.
- In fact: $R \leftrightarrows X$.
- If we *start* with X, we can rig it so the above condition above is satisfied.

Finding a Complete Pick NCP Kernel

The Setup

- M: a W^*-algebra;
- E: a W^*-correspondence over M
- X: an "admissible" sequence
- R: constructed from X, with sequence of weights, Z
- $\sigma : M \to \mathscr{B}(H)$ faithful, unital, W^*-homomorphism
- $\mathbb{D}(X,\sigma) = \left\{ \mathfrak{z} \in \mathcal{I}(\sigma^E \circ \varphi, \sigma) : \left\| \sum_{k=1}^{\infty} \mathfrak{z}^{(k)}(X_k \otimes I_H)\mathfrak{z}^{(k)*} \right\| < 1 \right\}$.
- $\mathfrak{z} \in \mathbb{D}(X,\sigma) \implies$ "ρ" extends to $(\sigma \times \mathfrak{z}) : \mathcal{H}^\infty(E,Z) \to \mathscr{B}(H)$.
- $Y \in \mathcal{H}^\infty(E,Z) \implies \widehat{Y} : \mathbb{D}(X,\sigma) \to \mathscr{B}(H)$

$$\widehat{Y}(\mathfrak{z}) = (\sigma \times \mathfrak{z})(Y)$$

- $\mathscr{R} : \mathbb{D}(X,\sigma) \times \mathbb{D}(X,\sigma) \to \mathscr{B}_{uw}(\sigma(M)', \mathscr{B}(H))$, (NCP kernel)

$$\mathscr{R}(\mathfrak{w},\mathfrak{z})(A) = \sum_{k=0}^{\infty} \mathfrak{w}^{(k)}(R_k^2 \otimes A)\mathfrak{z}^{(k)*}$$

Weighted Nevanlinna Pick

Theorem (G, 2014)

- $s, t \in \mathbb{N}$.
- $k \in \mathbb{N}$.
- k points, $\{\mathfrak{z}_i\}_{i=1}^k \subseteq \mathbb{D}(X, \sigma)$
- $\{B_i\}_{i=1}^k \subseteq M_{s\times s}(\mathscr{B}(H))$,
- $\{D_i\}_{i=1}^k \subseteq M_{s\times t}(\mathscr{B}(H))$.
- $\mathscr{A} : M_k(\sigma(M)') \to M_k(M_s(\mathscr{B}(H)))$

$$\mathscr{A}\left([a_{ij}]_{ij}^k\right) =$$

$$\left[B_i \begin{bmatrix} \mathscr{R}(\mathfrak{z}_i, \mathfrak{z}_j)(a_{ij}) & & \\ & \ddots & \\ & & \mathscr{R}(\mathfrak{z}_i, \mathfrak{z}_j)(a_{ij}) \end{bmatrix} B_j^* - D_i \begin{bmatrix} \mathscr{R}(\mathfrak{z}_i, \mathfrak{z}_j)(a_{ij}) & & \\ & \ddots & \\ & & \mathscr{R}(\mathfrak{z}_i, \mathfrak{z}_j)(a_{ij}) \end{bmatrix} D_j^* \right]_{ij}^k$$

Then $\mathscr{A}$ is completely positive iff there exists $Y = [Y_{mn}]_{m=1}^{s}\,{}_{n=1}^{t} \in M_{s\times t}(\mathcal{H}^\infty(E, Z))$ such that $\|Y\| \leq 1$ and

$$B_i \circ \left[\widehat{Y_{mn}}(\mathfrak{z}_i)\right]_{m=1}^{s}\,{}_{n=1}^{t} = D_i \qquad 1 \leq i \leq k$$

Multiplier Theorem

Theorem (G, 2014)

- $s, t \in \mathbb{N}$.
- $\Phi : \mathbb{D}(X, \sigma) \to M_{s \times t}(\mathscr{B}(H))$ any function.

Then Φ is an (s, t) multiplier of norm at most 1 iff there exists $Y = [Y_{mn}]_{m=1}^{s}{}_{n=1}^{t} \in M_{s \times t}(\mathcal{H}^\infty(E, Z))$ such that $\|Y\| \leq 1$ and

$$\Phi(\mathfrak{z}) = \left[\widehat{Y_{mn}}(\mathfrak{z})\right]_{m=1}^{s}{}_{n=1}^{t} \qquad \forall \mathfrak{z} \in \mathbb{D}(X, \sigma)$$

Corollary 1

If $Y \in \mathcal{H}^\infty(E, Z)$ then $\widehat{Y}$ is a multiplier of the *RKWC*, and *every* multiplier is obtained in this fashion.
(s = t = 1)

Corollary 2

$\mathscr{R}$ is an NCP Complete Pick Kernel. Whoohoo!

Is this the "Right" Complete Pick Property?

ANOTHER Equivalent Condition (McCullough-Quiggin)

An irreducible kernel, $\mathscr{K}$, is a Complete Pick kernel iff $\forall N \in \mathbb{N}$ and distinct $\{\omega_i\}_{i=1}^N \subset \Omega$,

$$F_N = \left[1 - \frac{k_{iN} k_{Nj}}{k_{ij} k_{NN}}\right]_{i,j=1}^{N-1} \geq 0 \qquad \text{where}$$

$$k_{ij} = \mathscr{K}(\omega_i, \omega_j) \in \mathbb{C}$$

One of These Things Is Not Like the Other...

$N \in \mathbb{N}$, $\{\mathfrak{w}_i\}_{i=1}^N \subseteq \Omega$, $\{A_i\}_{i=1}^{N-1} \subseteq \sigma(M)'$

$$\left[A_i A_j^* - \mathscr{R}_{ij}^{-1}\left(\mathscr{R}_{iN}(A_i)(\mathscr{R}_{NN}(I))^{-1}\mathscr{R}_{Nj}(A_j^*)\right)\right]_{i,j}^{N-1}$$

$$\mathscr{R}_{ij} = \mathscr{R}(\omega_i, \omega_j) \in CB(\sigma(M)')$$

An "Inner Multiplier"

- Ω: a set
- M, N: W^*-algebras
- E: an (Ω, M, N) RKWC

The Old

$\phi : \Omega \to N$ is an **"Outer multiplier" of** E if $\phi \cdot f \in E$, $\forall f \in E$,

$$\phi \cdot f(a, \omega) = \phi(\omega) f(a, \omega)$$

The New

$\mu : \Omega \to CB(M)$ is an **"Inner multiplier" of** E if $\mu \cdot f \in E$, $\forall f \in E$,

$$\mu \cdot f(a, \omega) = f(\mu(\omega)(a), \omega)$$

Inner (s, t) Multipliers

Inner (s, t) Multiplier

Given a family, $\{\mu_{mn}\}_{m=1}^{s}{}_{n=1}^{t}$ of inner multipliers, define $\mathscr{M} : \Omega \to \mathscr{F}(M_{s\times s}(M), M_{s\times t}(M))$

$$\mathscr{M}(\mathfrak{z})\left([a_{mn}]_{mn=1}^{s}\right) = \left[\sum_{p=1}^{s}\left(\mu_{pn}(\mathfrak{z})(a_{mp})\right)^{*}\right]_{m=1}^{s}{}_{n=1}^{t}$$

(an **Inner** (s, t) **Multiplier**).

The "Outer" W^*-$M_{s\times t}$ Pick Property, $s, t \in \mathbb{N}$ (G)

$\mathscr{K} : \Omega \times \Omega \to \mathscr{B}_{uw}(M, N)$, NCP kernel, has the $M_{s\times t}$ **Pick Property** if whenever

- $k \in \mathbb{N}$,
- $\{\omega_i\}_{i=1}^k \subseteq \Omega$,
- $\{B_i\}_{i=1}^k \subseteq M_{s\times s}(N)$,
- $\{D_i\}_{i=1}^k \subseteq M_{s\times t}(N)$
- $\mathscr{A} : M_k(M) \to M_k(M_s(N))$ is completely positive, where

$$\mathscr{A}\left([a_{ij}]_{ij}^k\right) =$$

$$\left[B_i \begin{bmatrix} \mathscr{K}(\mathfrak{w}_i, \mathfrak{w}_j)(a_{ij}) & & \\ & \ddots & \\ & & \mathscr{K}(\mathfrak{w}_i, \mathfrak{w}_j)(a_{ij}) \end{bmatrix} B_j^* - D_i \begin{bmatrix} \mathscr{K}(\mathfrak{w}_i, \mathfrak{w}_j)(a_{ij}) & & \\ & \ddots & \\ & & \mathscr{K}(\mathfrak{w}_i, \mathfrak{w}_j)(a_{ij}) \end{bmatrix} D_j^* \right]_{ij}^k$$

there exists an (s, t) multiplier, Φ, of norm ≤ 1 such that

$$B_i \Phi(\omega_i) = D_i, \quad 1 \leq i \leq k.$$

A Different Complete Pick Property:

Definition: The "Inner" W^*-$M_{s\times t}$ Pick Property, $s, t \in \mathbb{N}$ (G)

$\mathscr{K} : \Omega \times \Omega \to \mathscr{B}_{uw}(M, N)$, NCP kernel, has the **"Inner"** $M_{s\times t}$ **Pick Property** if whenever

- $k \in \mathbb{N}$
- $\{\mathfrak{z}\}_{i=1}^{k} \subseteq \Omega$
- $\{B_i\}_{i=1}^{k} \subseteq M_{s\times s}(M)$
- $\{D_i\}_{i=1}^{k} \subseteq M_{s\times t}(M)$
- $A = \left[\left(\mathscr{K}(\mathfrak{z}_i, \mathfrak{z}_j)\right)_s (B_i B_j^* - D_i D_j^*)\right]_{i,j=1}^{k} \geq 0$
  where $\left(\mathscr{K}(\mathfrak{z}_i, \mathfrak{z}_j)\right)_s : M_{s\times s}(M) \to M_{s\times s}(M)$ is given by

$$\left(\mathscr{K}(\mathfrak{z}_i, \mathfrak{z}_j)\right)_s([a_{mn}]_{mn}) = [\mathscr{K}(\mathfrak{z}_i, \mathfrak{z}_j)(a_{mn})]_{mn}$$

There exists an inner $s \times t$ multiplier, $\mathscr{M} : \Omega \to \mathscr{F}(M_{s\times s}(M), M_{s\times t}(M))$ such that

- $\mathscr{M}(\mathfrak{z}_i)(B_i) = D_i$ whenever $1 \leq i \leq k$
- $\|\mathscr{M}\| \leq 1$

Future Work

Definition: The "Inner" Complete Pick Property, $s, t \in \mathbb{N}$ (G)

A NCP kernel, $\mathscr{K} : \Omega \times \Omega \to \mathscr{B}_{uw}(M, N)$ has the **Inner W^*-Complete Pick Property** if it has the Inner W^*-$M_{s\times t}$ Pick Property $\forall s, t \in \mathbb{N}$.

Recall:

... IF there exists a completely positive map, $\mathscr{X}_{\mathfrak{z}} : \sigma(M)' \to \sigma(M)'$ such that $\|\mathscr{X}_{\mathfrak{z}}\| \leq 1$, and

$$\mathscr{R}_{\mathfrak{z}} = (\iota - \mathscr{X}_{\mathfrak{z}})^{-1},$$

then ρ extends to an ultraweakly continuous representation of $\mathcal{H}^\infty(E, Z)$

A New Toy to Play With

Hypothesis:
$\mathscr{R} : \Omega \times \Omega \to CB_{uw}(\sigma(M)')$ has the **Inner W^*-Complete Pick Property** if and only if there is an NCP kernel, $\mathscr{X} : \Omega \times \Omega \to CC_{uw}(\sigma(M)')$ such that $\forall \mathfrak{w}, \mathfrak{z} \in \Omega$.

$$\mathscr{R}(\mathfrak{w}, \mathfrak{z}) = (\iota - \mathscr{X}(\mathfrak{w}, \mathfrak{z}))^{-1},$$

Thanks!

Further Reading

- Jim Agler and John E. McCarthy.
  Pick interpolation and Hilbert function spaces, volume 44 of *Graduate Studies in Mathematics*.
  American Mathematical Society, Providence, RI, 2002.
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  PhD thesis, The University of Iowa, 2010.
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  Matricial function theory and weighted shifts.
  In preparation.