Traces and abelian core of graph algebras

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Outline

Tracial states

If A is a C^*-algebra then a *tracial state* on A is a state such that $\phi(xy) = \phi(yx)$ for all $x, y \in A$. The collection $T(A)$ of all tracial states on A has been studied for various classes of C^*-algebras.

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1. If $A = C(X) \rtimes_\alpha \mathbb{Z}$ for X a compact Hausdorff space, then tracial states are governed by α-invariant states on $C(X)$ (that is, α-invariant probability measures on X).
2. If A is an AF algebra then tracial states are governed by positive homomorphisms $K_0(A) \to \mathbb{R}$. (States on K-theory)
3. If $A = C^*(E)$ is a Cuntz-Krieger graph algebra then the tracial states of A correspond with the collection $T(E)$ of so-called graph traces on E.

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Most of these aren't bijections–in general an α-invariant state on $C(X)$ has various extensions to a tracial state on $C(X) \rtimes_\alpha \mathbb{Z}$, a graph trace induces various tracial states, etc.

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Our goal is to describe the tracial states on a C^*-algebra in terms of states on a certain distinguished maximal abelian subalgebra. We are most interested in the case where A is a C^*-algebra and $B \subset A$ is a certain type of maximal abelian subalgebra called an abelian core. In this talk I'll describe how to construct tracial states on graph algebras via their abelian cores.

Abelian core

The following definition was introduced in [?].

Definition

Suppose that A is a C^*-algebra and that $B \subset A$ is an abelian C^*-subalgebra. Then B is called an *abelian core* (for A) if the following conditions are satisfied:

1. there is a unique conditional expectation $P : A \to B$
2. the conditional expectation P is faithful ($P(a^*a) = 0$ implies $a = 0$).
3. B is a maximal abelian subalgebra of A
4. a $*$-representation $\pi : A \to B(H)$ is injective whenever its restriction to B is injective.

Abelian core

Such subalgebras are useful for studying states and representations because if $B \cong C_0(X)$, then each pure state of B is given by an evaluation map ev_x for some $x \in X$. In many cases the spectrum of the subalgebra is available to start with (i.e. $C(X)$ in a crossed product) or else is accessible through direct analysis (as we will see with graph algebras).

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Example

The Kadison-Singer problem was to show that if A is $\mathcal{B}(\ell^2(\mathbb{N}))$ and $B = \mathcal{D}(\ell^2(\mathbb{N}))$ is the maximal abelian subalgebra of diagonal operators, then B has the extension property. (Proved by Marcus-Spielman-Srivastava.)

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Theorem

If A a C^*-algebra and $B \subset A$ is a maximal abelian subalgebra, then the following are equivalent:

1. *B has the extension property relative to A*
2. *$A = B \oplus \overline{\text{span}}[A, B]$ (direct sum of closed subspaces), where $[A, B]$ is the set of elements of the form $ab - ba$ with $a \in A, b \in B$.*

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In particular, this shows that a two tracial states which agree on a maximal abelian subalgebra with the extension property must agree globally.

The Almost Extension Property

Definition

Let B be a C^*-subalgebra of the C^*-subalgebra A. The set of pure states on B with unique extension to A is denoted $P_1(B \uparrow A)$. We say that B has the *almost extension property* (AEP) if $P_1(B \uparrow A)$ is weak-$*$ dense in $P(B)$.

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Example

Let suppose that α is an action of $\mathbb{Z}$ on $C(X)$. Then $C(X) \subset C(X) \rtimes_\alpha \mathbb{Z}$ has the extension property iff the action is free. It has the almost extension property iff the set of aperiodic points is dense in X (that is, if α is a *topologically free* action).

Directed graphs

Definition

A directed graph $E = (E^0, E^1, r, s)$ consists of a countable set of vertices E^0, a countable set of edges E^1, and range and source maps $r, s : E^1 \to E^0$. A finite path in E is a sequence of edges $e_1 e_2 \dots e_k$ such that $s(e_i) = r(e_{i+1})$. (*Raeburn edge convention*). The collection of finite paths is denoted E^*. The collection of infinite paths is denoted by E^∞. A source is a vertex which is the range of no edge.

Graph algebras

Given a directed graph E we can define its graph algebra $C^*(E)$: this is the universal C^*-algebra generated by partial isometries $\{s_e : e \in E^1\}$ and projections $\{p_v : v \in E^0\}$ satisfying the following *Cuntz-Krieger relations*:

1. $s_e^* s_e = p_{s(e)}$;
2. $s_e s_e^* \perp s_f s_f^*$ if $e \neq f$;
3. $s_e s_e^* \leq p_{r(e)}$
4. if v receives a finite positive number of edges, then $p_v = \sum_{r(e)=v} s_e s_e^*$.

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These relations can be used to show that $C^*(E) = \overline{span}\{s_\lambda s_\mu^* : \lambda, \mu \in E^*\}$, where for $\lambda = e_1 \dots e_n$ we set $s_\lambda = s_{e_1} \dots s_{e_n}$.

The abelian core of a graph algebra

Definition

Let $M(E)$ be the C^*-subalgebra of $C^*(E)$ generated by

$$G_M(E) = \{s_\nu s_\nu^* : \nu \in E^*\} \cup \{s_\mu s_\lambda s_\mu^* : \lambda \text{ a loop without entry}\}.$$

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It is shown in [**?**] that $M(E)$ actually is an abelian core in the previous sense, i.e. that $M(E)$ is a maximal abelian subalgebra, it is the range of a unique faithful conditional expectation, and that a $*$-representation of $C^*(E)$ is faithful if and only if its restriction to $M(E)$ is faithful.

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$$P(s_\lambda s_\mu^*) = \begin{cases} s_\lambda s_\mu^*, & s_\lambda s_\mu^* \in G_M(E) \\ 0 & \text{else} \end{cases}.$$

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Theorem

Let E be a directed graph in which every loop has an entry and let $\pi : C^*(E) \to B$ be a $*$-homomorphism, where B is a C^*-algebra. Then π is isometric if and only if $\pi(p_v) \neq 0$ for every $v \in E^0$.

Note that this theorem puts a restriction on the graph.

Uniqueness theorems

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Let $\pi : C^*(E) \to B$ be a $*$-homomorphism which intertwines the gauge action on $C^*(E)$ with a continuous $\mathbb{T}$-action β on $\operatorname{im}\pi$, i.e. $\pi \circ \gamma_z = \beta_z \circ \pi$ for all $z \in \mathbb{T}$. Then π is isometric if and only if $\pi(p_v) \neq 0$ for every $v \in E^0$.

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Note that this theorem puts a restriction on the $*$-homomorphism.

Uniqueness Theorem

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Theorem

Let E be a directed graph and let $\pi : C^*(E) \to B$ be a $*$-homomorphism into a C^*-algebra. Then π is isometric if and only if $\pi(p_v) \neq 0$ for each $v \in E^0$ and for each cycle λ without entry the spectrum $\sigma(\pi(s_\lambda))$ contains $\mathbb{T}$.

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Let E be a directed graph and let $\pi : C^*(E) \to B$ be a $*$-homomorphism. If $\pi|_{M(E)}$ is isometric, then π is isometric.

Tracial states on graph algebras

The Cuntz-Krieger relations force tracial states to vanish on certain vertices. Here we extend the definition of *entry* to the loop $\lambda = e_1 e_2 \ldots e_n$ to mean a path $\mu = f_1 \ldots f_m$ such that $r(\mu) = r(e_k)$ and $f_1 \neq e_k$

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Lemma

Let v be a vertex in E which is the source of an entry to a loop. Then $\tau(p_v) = 0$ for any tracial state τ on $C^*(E)$.

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Lemma

Let v be a vertex in E which is the source of an entry to a loop. Then $\tau(p_v) = 0$ for any tracial state τ on $C^*(E)$.

This means that tracial states "live" on a certain collection of vertices. We use a familiar method to quotient out by this set of vertices.

Sealing a graph

Definition

A set H of vertices is *hereditary* if $r(e) \in H$ implies that $s(e) \in H$ for any $e \in E^1$. A set H of vertices is *saturated* if whenever a vertex w receives a finite positive number of edges, each with respective source vertex in H, then w must also belong to H. The *saturation* of a set H of vertices is the smallest saturated subset of E^0 containing H.

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The saturated and hereditary subsets of E^0 relate to ideals in a precise way. If H is a saturated and hereditary subset of E^0, then we define the ideal I_H to be the ideal of $C^*(E)$ generated by $\{p_v : v \in H\}$.

Sealed subgraph

Definition

Let E be a directed graph. Let H denote the subset of E^0 consisting of those vertices which emit entrances into loops. Let $\overline{H}$ denote the saturation of H. Then $E_s = E \setminus \overline{H} = (E^0 \setminus \overline{H}, s^{-1}(E^0 \setminus \overline{H}), r, s)$ is called the *sealed subgraph* of E.

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The sealed subgraph is *not* the largest subgraph in which no loop has an entry, but it is the largest such subgraph which can be seen as a quotient in the appropriate sense.

There is a homomorphism $\pi : C^*(E) \to C^*(E_s)$ given by taking the quotient by $I_{\overline{H}}$.

Theorem

Let E be a directed graph and let $\pi : C^*(E) \to C^*(E_s)$ be the quotient by $I_{\overline{H}}$. The map $\tau \to \tau \circ \pi$ gives an affine isomorphism from $T(C^*(E_s))$ to $T(C^*(E))$.

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Thus when studying $T(C^*(E))$, we can assume that E is *sealed* in the sense that no loop has an entry.

Sealed graphs and the extension property

Theorem

Let E be a sealed graph. Then $M(E) \subset C^*(E)$ has the extension property.

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Sketch of proof.

First identify the states of $D(E) = \overline{span}\{s_\lambda s_\lambda^* : \lambda \in E^*\}$ with the set $E^{\leq\infty}$ of paths which are either infinite or emanate from a source. These all have unique extensions to $C^*(E)$. The pure states of $M(E)$ consist of all these states as well as pure states given by pairs $(z, (\lambda, \mu))$, where λ is a cycle without entry, μ is a path with source equal to $s(\lambda)$, and $z \in \mathbb{C}$. Each of these also has a unique extension to $C^*(E)$ as shown in [**?**], [**?**]. □

Constructing tracial states

The following result classifies tracial states on graph algebras via their restrictions to the abelian core.

Theorem

Let E be a sealed directed graph and let $P : C^*(E) \to M(E)$ denote the conditional expectation onto the abelian core. For a state ϕ on $M(E)$, $\phi \circ P$ is a tracial state on $C^*(E)$ if and only if for diagonal generators $s_\alpha s_\lambda s_\alpha^*$ or $s_\beta s_\beta^*$ we have $\phi(s_\alpha s_\lambda s_\alpha^*) = \phi(s_\lambda)$ and $\phi(s_\beta s_\beta^*) = \phi(s_\beta^* s_\beta)$.

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The condition in the first part of the theorem can be reinterpreted as an equivariance condition for a measure on the Gelfand spectrum of the abelian core.

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2. Extend to k-graph Λ. Difficult to say what the analogue of sealed graph is. The "regular" infinite paths in Λ govern the relevant states and they are somewhat hard to control.
3. Try to find general conditions on an arbitrary abelian core $B \subset A$ such that tracial states on A are governed by suitably "A-invariant" states on B.

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Thank you!