# AF-Embeddings of Certain Graph $C^*$-Algebras
# Nebraska–Iowa Functional Analysis Seminar

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# Finiteness Properties

## Definition

A (unital) $C^*$-algebra $A$ is *finite* if whenever $v \in A$, we have $v^*v = 1$ implies $vv^* = 1$.

We say $A$ is *stably finite* if $M_n(A)$ is finite for all $n \geq 1$.

## Definition

A $C^*$-algebra $A$ is *quasidiagonal* if there are completely positive contractive maps $\varphi_n : A \to \mathbb{M}_{k(n)}$ such that for every $a, b \in A$,

$$\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0 \quad \text{and} \quad \|\varphi_n(a)\| \to \|a\|.$$

## Definition

A $C^*$-algebra $A$ is AF if there are finite dimensional subalgebras $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots A$ such that $\bigcup F_n$ is dense in $A$.

# Finiteness Properties

**Theorem**

AF-embeddable $\Rightarrow$ quasidiagonal $\Rightarrow$ stably finite.

The converses are false:

$C^*(\mathbb{F}_2)$ is quasidiagonal but not AF-embeddable.

$C^*_r(\mathbb{F}_2)$ is stably finite but not quasidiagonal.

**Conjectures (Blackadar, Kirchberg - 1997)**

Quasidiagonal and exact $\Rightarrow$ AF-embeddable.
Stably finite and nuclear $\Rightarrow$ quasidiagonal.

**Remark:** The conjectures are open even for group $C^*$-algebras and Type I $C^*$-algebras.

# Positive Results

For the following classes of $C^*$-algebras, stably finite $\Rightarrow$ AF-embeddable:

- (Pimsner, 1983) $C(X) \rtimes \mathbb{Z}$, where $X$ is a compact Hausdorff space;
- (Brown, 1998) $A \rtimes \mathbb{Z}$, where $A$ is an AF-alegbra;
- (S., 2014) $C^*(E)$ where either
  - $E$ is a discrete graph,
  - $E$ is a compact topological graph with no sinks, or
  - $E$ is a totally disconnected topological graph.
- (S., 2014) $\mathcal{O}_A(H)$ where $A$ is an AF-algebra and $H$ is a $C^*$-correspondence over $A$.

### Definition

A directed graph is a quadruple $E = (E^0, E^1, r, s)$ where $E^i$ are sets and $r, s : E^1 \to E^0$ are functions called the *range* and *source*.

### Definition

A *path* in $E$ is a word $\alpha = e_n \cdots e_2 e_1$ with $r(e_i) = s(e_{i+1})$ for every $i = 1, \ldots, n-1$. Set $s(\alpha) = s(e_1)$ and $r(\alpha) = r(e_n)$.

$$r(\alpha) \xleftarrow{e_n} \bullet \xleftarrow{e_{n-1}} \cdots \xleftarrow{e_3} \bullet \xleftarrow{e_2} \bullet \xleftarrow{e_1} s(\alpha)$$

### Definition

A *loop* in $E$ is a path $\alpha$ in $E$ such that $s(\alpha) = r(\alpha)$. We say $\alpha = e_n e_{n-1} \ldots e_1$ *has an entrance* if $|r^{-1}(r(e_i))| > 1$ for some $i = 1, \ldots, n$.

**Definition**

If $E$ is a directed graph, a Cuntz-Krieger $E$-family in a $C^*$-algebra $A$ is a collection of a pairwise orthogonal projections $(p_v)_{v \in E^0}$ and partial isometries $(s_e)_{e \in E^1}$ such that for all $v \in E^0$ and $e, f \in E^1$

1. $s_e^* s_e = p_{s(e)}$
2. $s_e s_e^* \leq p_{r(e)}$
3. $s_e^* s_f = 0$ if $e \neq f$
4. $\displaystyle\sum_{r(e)=v} s_e s_e^* = p_v$ if $0 < |r^{-1}(v)| < \infty$.

The universal $C^*$-algebra generated by a Cuntz-Krieger $E$-family is denoted $C^*(E)$.

# Examples

1. $E: \quad v \circlearrowright e$ $\qquad p_v = 1 \qquad s_e^* s_e = 1 \qquad s_e s_e^* = 1.$

$$C^*(E) \cong C(\mathbb{T})$$

2. $F: \quad e \circlearrowleft v \circlearrowright f$ $\qquad s_e^* s_e = s_f^* s_f = 1 \qquad s_e s_e^* + s_f s_f^* = 1.$

$$C^*(F) \cong \mathcal{O}_2$$

3. $G: \quad v \xrightarrow{e} w \circlearrowright f$ $\qquad s_f^* s_f = p_w \quad s_e s_e^* + s_f s_f^* = p_w.$

   $p_w$ is infinite. In fact $C^*(G)$ is the Toeplitz algebra.

4. Up to Morita equivalence, all AF-algebras and all Kirchberg algebras with torsion free $K_1$ group are graph algebras.

**Theorem (S. - 2014)**

Suppose $E$ is a countable directed graph. Then the following are equivalent:

1. $C^*(E)$ is AF-embeddable;
2. $C^*(E)$ is quasidiagonal;
3. $C^*(E)$ is stably finite;
4. $C^*(E)$ is finite;
5. No loop in $E$ has an entrance.

We will show (5) $\Rightarrow$ (1).

# Proof of AF-Embeddability

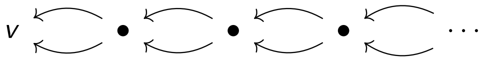
Suppose $E$ is a countable directed graph such that no loop in $E$ has an entrance.

The idea is to build a new graph $F$ such that $C^*(F)$ is AF and $C^*(E) \subseteq C^*(F)$.

$C^*(F)$ will be AF if and only if $F$ has no loops.

Each entry-less loop in a graph generates a copy of $C(\mathbb{T})$.

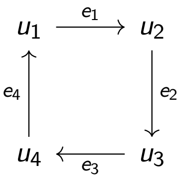
Let $B$ be the graph

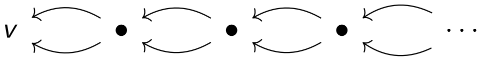



Note that $C^*(B) \cong \mathbb{M}_{2^\infty} \otimes \mathbb{K}$ and $p_v C^*(B) p_v \cong \mathbb{M}_{2^\infty}$.

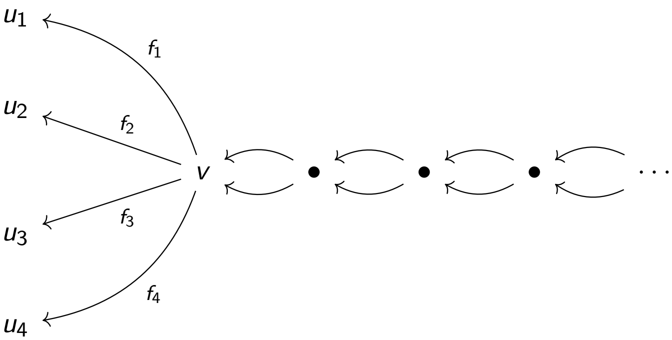
Let $t \in p_v C^*(E) p_v$ be a unitary with $\sigma(t) = \mathbb{T}$.

Given $E$ and $B$ as below:




form the graph $F$ below:



Given an entry-less loop $e_n \cdots e_2 e_1$ in $E$, define $\tilde{s}_e \in C^*(F)$ by

$$\tilde{s}_e = \begin{cases} s_e & e \in E^1 \setminus \{e_1, \ldots, e_n\} \\ s_{f_i} t s_{f_{i-1}}^* & e = e_i. \end{cases}$$

and define $\tilde{p}_w = p_w \in C^*(F)$ for $w \in E^0$.

Then $\{\tilde{p}_v, \tilde{s}_e\}$ is a Cuntz-Kreiger $E$-family (this uses that the loop $e_n \cdots e_2 e_1$ has no entrance).

There is an injective *-homomorphism $\varphi : C^*(E) \to C^*(F)$ given by $s_e \mapsto \tilde{s}_e$ and $p_v \mapsto \tilde{p}_v$ (use Szymański's Uniqueness Theorem).

Since $F$ has no loops, $C^*(F)$ is AF.

Hence $C^*(E)$ is AF-embeddable.

# Hilbert Modules

A Hilbert module over a $C^*$-algebra $A$ is a right $A$-module $H$ together with an *inner product* $\langle \cdot, \cdot \rangle : H \times H \to A$ such that

1. $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in H$,
2. $\langle \xi, \xi \rangle = 0$ implies $\xi = 0$ for $\xi \in H$.
3. $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*$ for $\xi, \eta \in H$,
4. $\langle \xi, \eta + \zeta a \rangle = \langle \xi, \eta \rangle + \langle \xi, \zeta \rangle a$ for $\xi, \eta, \zeta \in H$, $a \in A$,

and such that $H$ is complete in the norm $\|\xi\|_H^2 = \|\langle \xi, \xi \rangle\|_A$.

An operator $T : H \to H$ is called *adjointable* if there is a $T^* : H \to H$ such that

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle \qquad \text{for every } \xi, \eta \in H.$$

Every adjointable operator is bounded and $A$-linear.

The collection $\mathbb{B}(H)$ of all adjointable operators on $H$ forms a $C^*$-algebra.

A $C^*$-correspondence over $A$ is a Hilbert $A$-module $H$ together with a $*$-homomorphism $\lambda : A \to \mathbb{B}(H)$.

Note that $H$ is an $A$–$A$ bimodule with $a\xi := \lambda(a)(\xi)$ for $a \in A$ and $\xi \in H$.

**Example:** Suppose $A$ is a unital $C^*$-algebra and $\alpha : A \to A$ is a $*$-homomorphism.

Define $H_\alpha = A$ with the obvious Hilbert $A$-module structure and define the left action by $\lambda = \alpha : A \to A = \mathbb{B}(H_\alpha)$.

# Toeplitz Representations

Let $H$ be a $C^*$-correspondence over $A$. A *covariant Toeplitz representation* of $H$ on a $C^*$-algebra $B$ consists of

- a $*$-homomorphism $\pi : A \to B$,
- a linear map $\tau : H \to B$,

such that the following hold:

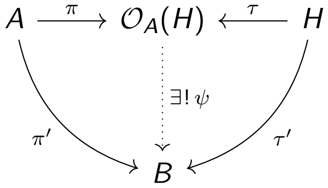
1. $\tau(a\xi) = \pi(a)\tau(\xi)$,
2. $\tau(\xi a) = \tau(\xi)\pi(a)$,
3. $\pi(\langle \xi, \eta \rangle) = \tau(\xi)^*\tau(\eta)$,
4. a certain covariance condition

**Remark:** If $\pi$ and $\tau$ are both contractive: Given $\xi \in H$,

$$\|\tau(\xi)\|^2 = \|\tau(\xi)^*\tau(\xi)\| = \|\pi(\langle \xi, \xi \rangle)\| \leq \|\langle \xi, \xi \rangle\| = \|\xi\|^2.$$

Moreover, if $\pi$ is injective, then $\pi$ and $\tau$ are isometric.

There is a $C^*$-algebra $\mathcal{O}_A(H)$ and a covariant Toeplitz representation $(\pi, \tau) : (A, H) \to \mathcal{O}_A(H)$ which is universal in the following sense:




Given another covariant Toeplitz representation $(\pi', \tau') : (A, B) \to B$, there is a unique $*$-homomorphism $\psi : \mathcal{O}_A(H) \to B$ such that $\pi' = \psi \circ \pi$ and $\tau' = \psi \circ \tau$.

# Example

Let $A$ be a $C^*$-algebra and let $\alpha : A \to A$ be an automorphism.

Define $H_\alpha = A$ with the obvious right $A$-module structure and define the left action by $\lambda = \alpha : A \to A \subseteq \mathbb{B}(H_\alpha)$.

Then $\mathcal{O}_A(H_\alpha) \cong A \rtimes_\alpha \mathbb{Z}$.

In general, the algebra $\mathcal{O}_A(H)$ is thought of as the crossed product of $A$ by the generalized morphism $H$.

In fact, there is a certain "skew–product" correspondence $H^\infty$ over a $C^*$-algebra $A^\infty$ such that

$$\mathcal{O}_A(H) \otimes \mathbb{K} \cong \mathcal{O}_{A^\infty}(H^\infty) \rtimes_\sigma \mathbb{Z} \quad \text{and} \quad \mathcal{O}_{A^\infty}(H^\infty) \cong \mathcal{O}_A(H) \rtimes_\gamma \mathbb{T}.$$

# $C^*$-correspondences over AF-algebras

**Theorem**

*If $H$ is a (non-degenerate) $C^*$-correspondence over an AF-algebra $A$, then $\mathcal{O}_A(H) \rtimes_\gamma \mathbb{T}$ is AF, where $\gamma$ is the gauge action on $\mathcal{O}_A(H)$.*

**Corollary**

*If $H$ is a $C^*$-correspondence over an AF-algebra $A$, then $\mathcal{O}_A(H)$ is Morita equivalent to a crossed product of an AF-algebra by $\mathbb{Z}$.*

*Hence the following are equivalent:*

1. *$\mathcal{O}_A(H)$ is AF-embeddable;*
2. *$\mathcal{O}_A(H)$ is quasidiagonal;*
3. *$\mathcal{O}_A(H)$ is stably finite.*

# Topological Graphs

A topological graph $E = (E^0, E^1, r, s)$ is a directed graph such that the $E^i$ are locally compact Hausdorff spaces, $r : E^1 \to E^0$ is continuous, and $s : E^1 \to E^0$ is a local homeomorphism.

$C_c(E^1)$ is a bimodule over $C_0(E^0)$ with

$$(a\xi)(e) = a(r(e))\xi(e) \qquad (\xi a)(e) = \xi(e)a(s(e))$$

for $a \in C_0(E^0)$, $\xi \in C_c(E^1)$ and $e \in E^1$.

Define an $C_0(E^0)$-valued inner product on $C_c(E^1)$ by

$$\langle \xi, \eta \rangle (v) = \sum_{s(e)=v} \xi(e)\overline{\eta(e)} \qquad \xi, \eta \in C_c(E^1),\ v \in E^0.$$

Complete to get a $C^*$-correspondence and let $C^*(E)$ denote the Cuntz-Pimsner algebra.

# Examples

1. Suppose $X$ is a locally compact, Hausdorff space and $\sigma$ is a homeomorphism of $X$.

   Set $E^0 = E^1 = X$, $r = \text{id}$, and $s = \sigma$.

   $E$ is a topological graph and $C^*(E) \cong C_0(X) \rtimes_\sigma \mathbb{Z}$

2. In general, if $E$ is a topological graph, $E^1$ can be thought of as a partially defined, multi-valued, continuous map $E^0 \to E^0$ given by $v \mapsto r(s^{-1}(v))$.

   $C^*(E)$ is thought of as the crossed product of $C_0(E^0)$ by $E^1$.

3. Topological graph algebras include many "classifiable" $C^*$-algebras; e.g., AF-algebra, Kirchberg algebras, and simple $A\mathbb{T}$-algebras with real rank zero.

**Theorem**

*Suppose $E$ is a compact topological graph with no sinks. The following are equivalent:*

1. *$C^*(E)$ is AF-embeddable;*
2. *$C^*(E)$ is quasidiagonal;*
3. *$C^*(E)$ is stably finite;*
4. *$C^*(E)$ is finite;*
5. *Every vertex emits exactly one edge and every vertex is "pseudo-periodic" in the sense of Pimsner.*

*In this case, $C^*(E) \cong C(E^\infty) \rtimes_\sigma \mathbb{Z}$, where $\sigma$ is the shift on $E^\infty$.*

# Outline of Proof

The proof relies heavily on results from
Z. Jabłoński, I. Jung, J. Stochel, Weighted shifts on directed trees, *Mem. Amer. Math. Soc.*, **216**(2012), no. 1017.

Given a directed tree $\Gamma = (\Gamma^0, \Gamma^1, r, s)$ and $\lambda = (\lambda_v)_{v \in \Gamma^0}$, we define weighted shifts $S_\lambda$ on $\ell^2(\Gamma^0)$ by

$$S_\lambda \delta_v = \sum_{s(e)=v} \lambda_{r(e)} \delta_{r(e)} \qquad \text{for } \delta_v \in \mathcal{D}(S_\lambda).$$

**Remark:** $S_\lambda$ is everywhere defined and bounded if and only if

$$\sup_{v \in \Gamma^0} \sum_{s(e)=v} |\lambda_{r(e)}|^2 < \infty.$$

All the shift operators we consider will be bounded.

# Examples

$$\cdots \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$$

Then $S_\lambda$ is a weighted bilateral shift. In particular,

$$S_\lambda \delta_n = \lambda_{n+1} \delta_{n+1}.$$

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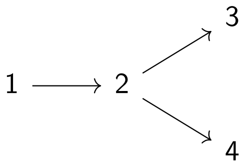


$$S_\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & \lambda_4 & 0 & 0 \end{pmatrix}$$

# Fredholm Theory of Shifts

There is a very detailed analysis of the Fredholm theory of directed shifts from Jabłoński, Jung, and Stochel.

**Theorem**

*Suppose $S_\lambda$ is a bounded weighted shift of a tree $\Gamma$. Then $S_\lambda$ is bounded below if and only if*

$$\inf_{v \in E^0} \|S_\lambda \delta_v\| > 0.$$

**Theorem**

*Suppose $S_\lambda$ is bounded and bounded below. Then $S_\lambda$ is surjective if and only if every vertex in $\Gamma$ emits exactly one edge.*

# Outline of Proof

Suppose $E$ is a compact topological graph with no sinks and $C^*(E)$ if finite. We claim $E$ every vertex in $E$ has emits exactly one edge.

We may view $E^\infty$ as a directed graph $F$ where $F^0 = F^1 = E^\infty$, $r = \mathrm{id}$, and $s = \sigma$ is the backward shift.

Represent $C^*(E)$ faithfully on $\mathbb{B}(\ell^2(E^\infty))$ by

$$\pi : C(E^0) \to \mathbb{B}(\ell^2(E^\infty)) \qquad \pi(a)\delta_\alpha = a(r(\alpha))\delta(v)$$

$$\tau : C(E^1) \to \mathbb{B}(\ell^2(E^\infty)) \qquad \tau(\xi)\delta_\alpha = \sum_{s(e)=r(\alpha)} \xi(e)\delta_{e\alpha}.$$

Then $\tau(1)$ is an invertible weighted shift in $C^*(E) \subseteq \mathbb{B}(\ell^2(E^\infty))$ . So every vertex of $E^\infty$ admits exactly one edge, and hence every vertex of $E$ admits exactly one edge.

We have $C^*(E) \cong C(E^0) \rtimes_{r\circ s^{-1}} \mathbb{N} \cong C(E^\infty) \rtimes_\sigma \mathbb{Z}$.

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