On Spectra of a Cantor Measure

John Haussermann* and Dorin Ervin Dutkay

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Overview

Consider the Cantor set obtained from the interval $[0,1]$, dividing it into four equal intervals and keeping the first and the third, $[0,1/4]$ and $[1/2,3/4]$, and repeating the procedure.
This can be described in terms of iterated function systems: let

$$\tau_0(x) = x/4 \text{ and } \tau_2(x) = (x+2)/4, \quad (x \in \mathbb{R}).$$

The Cantor set X_4 is the unique compact set that satisfies the invariance condition

$$X_4 = \tau_0(X_4) \cup \tau_2(X_4).$$

The set X_4 is described also in terms of the base 4 decomposition of real numbers :

$$X_4 = \left\{ \sum_{k=1}^{n} 4^{-k} b_k : b_k \in \{0,2\}, n \in \mathbb{N} \right\}.$$

Overview

On the set X_4 one considers the Hausdorff measure μ of dimension $\log_4 2 = \frac{1}{2}$. In terms of iterated function systems, the measure μ is the invariant measure for the iterated function system, that is, the unique Borel probability measure that satsifies the invariance equation

$$\mu(E) = \frac{1}{2}\left(\mu(\tau_0^{-1}E) + \mu(\tau_2^{-1}E)\right), \text{ for all Borel sets } E \subset \mathbb{R}. \tag{0.1}$$

Equivalently, for all continuous compactly supported functions f,

$$\int f\,d\mu = \frac{1}{2}\left(\int f \circ \tau_0\,d\mu + \int f \circ \tau_2\,d\mu\right). \tag{0.2}$$

Overview

We denote, for $\lambda \in \mathbb{R}$:

$$e_\lambda(x) = e^{2\pi i\lambda\cdot x}, \quad (x \in \mathbb{R}).$$

The Hilbert space $L^2(\mu)$ has an orthonormal basis formed with exponential functions, i.e., a Fourier basis, $E(\Gamma_0) := \{e_\lambda : \lambda \in \Gamma_0\}$ where

$$\Gamma_0 := \left\{ \sum_{k=0}^{n} 4^k l_k : l_k \in \{0,1\}, n \in \mathbb{N} \right\}. \tag{0.3}$$

Definition

We say that the subset Γ of $\mathbb{R}$ is a *spectrum* for the measure μ if the corresponding family of exponential $E(\Gamma) := \{e_\lambda : \lambda \in \Gamma\}$ is an orthonormal basis for $L^2(\mu)$. We say that Γ is complete/incomplete if the set $E(\Gamma)$ is as such in $L^2(\mu)$.

Overview

Question

For what digits $\{0, m\}$ with $m \in \mathbb{N}$ odd is the set

$$\Gamma(m) := m\Gamma_0 = \left\{ \sum_{k=0}^{n} 4^k l_k : l_k \in \{0, m\}, n \in \mathbb{N} \right\}$$

a spectrum for $L^2(\mu)$?

Extreme Cycles

Definition

Let $m \in \mathbb{N}$ be an odd number. We say that a finite set $\{x_0, x_1, \ldots, x_{r-1}\}$ is an *extreme cyle* (for the digits $\{0, m\}$) if there exist $l_0, \ldots, l_{r-1} \in \{0, m\}$ such that

$$x_1 = \frac{x_0 + l_0}{4}, \quad x_2 = \frac{x_1 + l_1}{4}, \quad \ldots \quad ,$$

$$x_{r-1} = \frac{x_{r-2} + l_{r-2}}{4}, \quad x_0 = \frac{x_{r-1} + l_{r-1}}{4},$$

and

$$\left| \frac{1 + e^{2\pi i 2 x_k}}{2} \right| = 1, \quad (k \in \{0, \ldots, r-1\}). \tag{0.4}$$

The points x_i are called extreme cycle points.

Extreme Cycles

Theorem

Let $m \in \mathbb{N}$ be odd. The set $\Gamma(m)$ is a spectrum for the measure μ if and only if the only extreme cycle for the digit set $\{0, m\}$ is the trivial one $\{0\}$.

Extreme Cycles: Examples

Recall

$$x_1 = \frac{x_0 + l_0}{4}, \quad x_2 = \frac{x_1 + l_1}{4}, \quad \ldots \quad ,$$

$$x_{r-1} = \frac{x_{r-2} + l_{r-2}}{4}, \quad x_0 = \frac{x_{r-1} + l_{r-1}}{4},$$

where $l_j \in \{0, m\}$.
Let $m = 3$.

$$\frac{1+3}{4} = 1,$$

so $\{1\}$ is an extreme cycle for the digit set $\{0, 3\}$.
Let $m = 85$.

$$\frac{7+85}{4} = 23, \quad \frac{23+85}{27} = 1, \quad \frac{27+85}{4} = 28, \quad \frac{28+0}{4} = 7,$$

so $\{7, 23, 27, 28\}$ is an extreme cycle for the digit set $\{0, 85\}$.

Extreme Cycles

Lemma

If x_0 *is an extreme cycle point then* $x_0 \in \mathbb{Z}$, x_0 *has a periodic base 4 expansion*

$$x_0 = \frac{a_0}{4} + \frac{a_1}{4^2} + \cdots + \frac{a_{r-1}}{4^r} + \frac{a_0}{4^{r+1}} + \cdots + \frac{a_{r-1}}{4^{2r}} + \ldots, \tag{0.5}$$

with $a_k \in \{0, m\}$, *and* $0 \leq x_0 \leq \frac{m}{3}$.

Proposition

If $\Gamma(m)$ *is incomplete then* $\Gamma(km)$ *is incomplete for all* $k \in \mathbb{Z}$, $k \geq 1$.

Prime Powers

Theorem

Let $m > 3$ be an odd number not divisible by 3. Let $G = \{4^j \pmod m | j \in \mathbb{N}\}$. If any of the numbers $-1 \pmod m$, $-2 \pmod m$, or $2 \pmod m$, then $\Gamma(m)$ is complete.

Prime Powers

Assume for contradiction's sake that $\Gamma(m)$ is not spectral. Then there is a non-trivial extreme cycle $X = \{x_0, ..., x_{r-1}\}$ for the digit set $\{0, m\}$. From the relation between the cycle points,

$$x_{j+1} = \frac{x_j + b_j}{4}, \tag{0.6}$$

where $b_j \in \{0, m\}$, we have that $4x_{j+1} \equiv x_j (\bmod m)$.
Thus,

$$4^{r-k} x_0 \equiv x_0 (\bmod m, k \in \{0, \ldots, r\}), \tag{0.7}$$

so, for all $k \in \mathbb{N}$, the number $4^k x_0$ is congruent modulo m with an element of the extreme cycle X. But then, the hypothesis implies that there is a number $c \in \{-1, 2, -2\}$, so the number cx_0 is congruent modulo m with an element in X, and since x_0 is arbitrary in the cycle, we get that cx_j is congruent to an element in X for any j.

Prime Powers

In the following arguments we use the fact that since m is not divisible by 3, the condition on cycle points $0 \leq x_j \leq \frac{m}{3}$ implies $0 \leq x_j < \frac{m}{3}$.
If $c = -1$, then $-x_0 \pmod{m} \in X$. Since $x_0 < \frac{m}{3}$, $-x_0 \pmod{m} > \frac{m}{3}$, a contradiction.
If $c = -2$, then $-2x_0 \pmod{m} \in X$. Since $x_0 < \frac{m}{3}$, $-2x_0 \pmod{m} > \frac{m}{3}$, a contradiction.
If $c = 2$, then $2x_j \pmod{m} \in X$ for all j. Let x_N be the largest element of the extreme cycle. Since $x_N < \frac{m}{3}$, $2x_N \pmod{m} = 2x_N$. This number is in X, a contradiction to the maximality of x_N.

Prime Powers

Theorem

If p is a prime number, $p > 3$ and $n \in \mathbb{N}$, then $\Gamma(p^n)$ is complete.

It is well known that the equation $x^2 \equiv b \pmod{p^n}$ has zero or two solutions.
Let a be the smallest positive integer such that $4^a \equiv 1 \pmod{p^n}$.
If a is even, then we have $(4^{a/2})^2 \equiv 1 \pmod{p^n}$ so $4^{a/2} \equiv \pm 1 \pmod{p^n}$.
Since $4^{a/2} \not\equiv 1 \pmod{p^n}$ we get $4^{a/2} \equiv -1 \pmod{p^n}$.
If a is odd, then $(4^{\frac{a+1}{2}})^2 \equiv 4 \pmod{p^n}$. Therefore $4^{\frac{a+1}{2}} \equiv \pm 2 \pmod{p^n}$.
In both cases, the result follows from the previous Theorem.

Composite Numbers

Definition

We say that an odd number m is primitive if $\Gamma(m)$ is incomplete and, for all proper divisors d of m, $\Gamma(d)$ is complete. In other words, there exist extreme cycles for the digits $\{0, m\}$ and there are no extreme cycles for the digitis $\{0, d\}$ for any proper divisor d of m.
For an integer m, the order of 4 in the group $U(\mathbb{Z}_m)$ is the smallest positive integer a such that $4^a \equiv 1 \bmod m$. We denote a by $o_4(m)$, and the set of powers of 4 in $U(\mathbb{Z}_m)$ by G.

Composite Numbers

Proposition

Let m be a primitive number and let $C = \{x_0, \ldots, x_{p-1}\}$ be an extreme cycle. Then:

1. *The length p of the cycle is equal to $o_4(m)$.*
2. *Every element of the cycle x_j is mutually prime with m.*
3. *The extreme cycle C is a coset of the group G: $C = x_0 G$.*

Composite Numbers

Theorem

There are infinitely many primitive numbers.

Proposition

Let m and n be mutually prime odd integers. Then

$$o_4(mn) = \mathrm{lcm}(o_4(m), o_4(n)).$$

Definition

For a prime number $p \geq 3$, we say that p is *simple* if $o_4(p) < o_4(p^2)$.

Composite Numbers

Proposition

Let m be an odd number. If

$$o_4(m) > \sqrt{\frac{4m}{3}}$$

then m cannot be primitive.

Lemma

Let $a, b \geq 1$ be some odd numbers. Assume that $o_4(ab) > \frac{a}{3} o_4(b)$. Then ab cannot be primitive.

Composite Numbers

Corollary

Let $p_1, \ldots, p_r$ *be distinct prime numbers strictly larger than 5. Assume the following conditions are satisfied:*

1. *The numbers* $o_4(p_1), \ldots, o_4(p_r), p_1, \ldots, p_r$ *are mutually prime.*
2. $o_4(p_g) = \frac{p_g - 1}{2}$ *for some* g*, and* p_g *is simple.*

Then the set $\Gamma(p_1^{k_1} \ldots p_r^{k_r})$ *is not primitive for any* $k_1 \geq 0, \ldots, k_r \geq 0$ *provided that* $k_g \geq 1$*.*

Composite Numbers

Corollary

Let $p_1, \dots, p_r$ *be distinct simple prime numbers strictly larger than 3. Assume the following conditions are satisfied:*

1. *The numbers* $o_4(p_1), \dots, o_4(p_r), p_1, \dots, p_r$ *are mutually prime.*
2. $o_4(p_j) > \sqrt{\sqrt{\frac{4}{3}p_j}}$ *for all* j.

Then the set $\Gamma(p_1^{k_1} \dots p_r^{k_r})$ *is complete for any* $k_1 \geq 0, \dots, k_r \geq 0$.

Thank you!