The Jacobson radical of semicrossed products of the disk algebra

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Disk Algebra

What is the disk algebra?

- Let $\mathbb{D}$ be the open unit disc and $\mathbb{T}$ the unit circle.
- The disk algebra is an algebra

$$\mathcal{A}(\mathbb{D}) = \{f : f \text{ is analytic on } \mathbb{D} \text{ and continuous on } \overline{\mathbb{D}}\}.$$

Finite Blaschke Products

- A finite Blaschke product is a function φ of the form:

$$\varphi(z) = u \prod_{i=1}^{N} \frac{z - a_i}{1 - \bar{a}_i z},$$

where $|u| = 1$, and $a_i \in \mathbb{D}$, $i = 1, 2, \ldots, N$.

- Define an *endomorphism* α of $\mathcal{A}(\mathbb{D})$ by

$$\alpha(f) = f \circ \varphi,$$

where $f \in \mathcal{A}(\mathbb{D})$ and φ is a finite Blaschke product.

Semicrossed products

- Let $\mathcal{P}$ be the algebra of formal polynomials
$$\sum U^k f_k$$
with $f_k \in \mathcal{A}(\mathbb{D})$ where U satisfies
$$fU = U\alpha(f)$$
for any $f \in \mathcal{A}(\mathbb{D})$.
- The semicrossed product $\mathcal{A}(\mathbb{D}) \times_\alpha \mathbb{Z}^+$ is the completion of $\mathcal{P}$ with respect to the following norm.

Semicrossed products

- Consider a pair (A, T) of contractions satisfying

$$AT = T\varphi(A).$$

- There is a contractive representation ρ of $\mathcal{A}(\mathbb{D})$ given by

$$\rho(f) = f(A).$$

- Define a representation of $\mathcal{P}$

$$\rho \times T(\sum U^k f_k) = \sum T^k \rho(f_k).$$

- Then the norm on $\mathcal{P}$ is defined by

$$||\sum U^k f_k|| = \sup_{(\rho, T) \text{ covariant}} ||\sum T^k \rho(f_k)||.$$

Fourier Coefficients

- For $t \in [0, 1]$, define γ_t acting on $\mathcal{P}$ by

$$\gamma_t(\sum U^k f_k) = \sum U^k e^{2\pi ikt} f_k$$

which extends to an automorphism of $\mathcal{A}(\mathbb{D}) \times_\alpha \mathbb{Z}^+$.

- Define the k^{th} Fourier coefficient $\pi_k(F)$ of F by

$$U^k \pi_k(F) = \int_{\mathbb{T}} e^{-2\pi ikt} \gamma_t(F) dm(t)$$

where dm is normalized Lebesgue measure on the unit circle.

- If F is in the polynomial $\mathcal{P}$, then $\pi_k(F) = f_k$ and $||\pi_k(F)|| \leq ||F||$.

Quasinilpotent Elements and Jacobson Radical

Definition

F is a quasinilpotent element if $\lim_{n\to\infty} ||F^n||^{\frac{1}{n}} = 0$.

Definition

The Jacobson radical of an operator algebra $\mathfrak{A}$ is the maximal left ideal of $\mathfrak{A}$ which is contained in the set of quasinilpotent elements, denoted by $\mathrm{Rad}(\mathfrak{A})$.

Previous Results

1. T. Hoover, J. Peters, and W. Wogen, *Spectral properties of semicrossed products,* Houston J. Math., *19* (1993), 649–660.
2. A. P. Donsig, A. Katavolos, and A. Manoussos, *The Jacobson radical for analytic crossed products,* J. Funct. Anal., *187* (2001), 129–145.
3. K. A. Davidson and E. Katsoulis, *Semicrossed products of the disk algebra,* preprint, arXiv:1104.1398v1, 2011.

Previous Results

Theorem (Donsig et al., 2001)

$Rad(C(\mathbb{T}) \times_\alpha \mathbb{Z}^+)$ is the set $\{F \in C(\mathbb{T}) \times_\alpha \mathbb{Z}^+ : \pi_0(F) = 0$, and $\pi_k(F)$ vanishes on the recurrent set$\}$.

Theorem (Davidson and Katsoulis, 2011)

Let φ be a finite Blaschke product, and let $\alpha(f) = f \circ \varphi$. Then $\mathcal{A}(\mathbb{D}) \times_\alpha \mathbb{Z}^+$ is a subalgebra of $C(\mathbb{T}) \times_\alpha \mathbb{Z}^+$.

Classification of Finite Blaschke Products

Definition

A finite Blaschke product φ is said to be

1. elliptic if there exists a fixed point in $\mathbb{D}$;
2. hyperbolic if there exists the Denjoy-Wolff point $z_0 \in \mathbb{T}$ such that $\varphi'(z_0) < 1$;
3. parabolic if there exists the Denjoy-Wolff point $z_0 \in \mathbb{T}$ such that $\varphi'(z_0) = 1$.

Julia Sets

Definition

The Fatou set $\mathcal{F}$ of φ is the set of points z in $\overline{\mathbb{C}}$ such that $\{\varphi^n\}$ is a normal family in some neighborhood of z. The Julia set is the complement of the Fatou set.

Remark

1. The Julia set is the closure of the repelling periodic points.
2. The Julia set of a finite Blaschke product is either $\mathbb{T}$ or a Cantor set of $\mathbb{T}$.

Hyperbolic Distance

Definition

The hyperbolic distance d in $\mathbb{D}$ is defined by

$$d(z,w) = \log \frac{1 + \frac{|z-w|}{|1-\bar{z}w|}}{1 - \frac{|z-w|}{|1-\bar{z}w|}}$$

where $z, w \in \mathbb{D}$.

Definition

A finite Blaschke product φ is said to be

1. of zero hyperbolic step if $\lim_{n\to\infty} d(\varphi^n(z), \varphi^{n+1}(z)) = 0$ for some $z \in \mathbb{D}$.
2. of positive hyperbolic step otherwise.

Density of Recurrent Points

Theorem (Basallote et al., 2009, Contreras et al., 2007 and Hamilton, 1996)

φ is of zero hyperbolic step if and only if its Julia set is $\mathbb{T}$ if and only if it is ergodic.

Remark

Here we assume φ is ergodic without assuming that φ is measure-preserving.

Hyperbolic Step

Theorem (Basallote et al., 2009)

Let φ be a finite Blaschke product.

1. *If φ is elliptic, then it is of zero hyperbolic step.*
2. *If φ is hyperbolic, then it is of positive hyperbolic step.*

Theorem (Contreras et al., 2007)

If φ is a parabolic finite Blaschke product with Denjoy-Wolff point 1, then φ is of zero hyperbolic step if and only if $\varphi''(1) = 0$ if and only if $\sum_{i=1}^{N} \frac{1 - |a_i|^2}{|1 - a_i|^2} Im(a_i) = 0$ where a_i's are as in (2).

Topological Transitivity

Lemma

If φ is elliptic or parabolic with zero hyperbolic step, then the recurrent points of φ are dense in $\mathbb{T}$.

Theorem

If φ is elliptic or parabolic with zero hyperbolic step, then φ is topologically transitive.

The Recurrent Set

Theorem (Contreras et al., 2007)

If φ is hyperbolic or parabolic with positive hyperbolic step, then $(\varphi^n(z))$ converges to the Denjoy-Wolff point of φ, for almost every $z \in \mathbb{T}$.

Theorem

If φ is hyperbolic or parabolic with positive hyperbolic step, then the closure of the set of recurrent points of φ is the union of the Julia set and the Denjoy-Wolff point.

Quasinilpotent Elements

Lemma

1. *Let* $F \in A(\mathbb{D}) \times_\alpha \mathbb{Z}^+$. *If* $\pi_0(F)$ *is not identically zero or* $\pi_k(F)$ *does not vanish on the fixed point set of* φ *for some* $k > 0$, *then* F *is not quasinilpotent.*
2. *Let* $f \in \mathcal{A}(\mathbb{D})$. *If* f *does not vanish on the set of recurrent points of* φ, *then there is an* $n \in \mathbb{N}$ *such that* $U^n f$ *is not quasinilpotent.*

The Jacobson Radical of the Semicrossed Product

Theorem

Rad($\mathcal{A}(\mathbb{D}) \times_\alpha \mathbb{Z}^+$) is the set $\{F \in \mathcal{A}(\mathbb{D}) \times_\alpha \mathbb{Z}^+ : \pi_0(F) = 0$, and $\pi_k(F)$ vanishes on the set of recurrent points of φ, for $k > 0\}$.

Proof.

- From Donsig's result, if F is such that $\pi_0(F) = 0$ and $\pi_k(F)$ vanishes on the recurrent set for $j > 0$, then F is in Rad($C(\mathbb{T}) \times_\alpha \mathbb{Z}^+$).
- $I = \text{Rad}(C(\mathbb{T}) \times_\alpha \mathbb{Z}^+) \cap \mathcal{A}(\mathbb{D}) \times_\alpha \mathbb{Z}^+$ is contained in Rad($\mathcal{A}(\mathbb{D}) \times_\alpha \mathbb{Z}^+$).
- Let $F \in \text{Rad}(\mathcal{A}(\mathbb{D}) \times_\alpha \mathbb{Z}^+)$.
- Then $\pi_0(F) = 0$ and $\pi_k(F)(z_0) = 0\ \forall k > 0$ where z_0 is the Denjoy-Wolff point.
- Suppose that not all f_i's vanish on $\mathcal{J}$.

Proof (continue).

- Then $\exists k$ so that $f_k(x_0) \neq 0$ and f_i's vanish on $\mathcal{J}$ $\forall i < k$ for some x_0 where $\varphi^n(x_0) = x_0$ for some n.
- Choose j so that $U^j(U^k f_k) = U^m f_k$ where m is a multiple of n.
- For $l > 0$,

$$\begin{aligned} ||(U^j F)^l|| &\geq ||\pi_{ml}((U^j F)^l)|| \\ &\geq |\pi_{ml}((U^j F)^l)(x_0)| \\ &= |f_k(x_0) f_k \circ \varphi^m(x_0) \dots f_k \circ \varphi^{(l-1)m}(x_0)| \\ &= |f_k(x_0)|^l. \end{aligned}$$

- Thus, $U^j F$ is not quasinilpotent, a contradiction.
- f_i must vanish on $\mathcal{J}$ for all i, and so $F \in I$.

□

Theorem

Rad($\mathcal{A}(\mathbb{D}) \times_{\alpha} \mathbb{Z}^{+}$) *is*

1. *nonzero if φ is hyperbolic or parabolic with positive hyperbolic step;*
2. *zero if φ is elliptic or parabolic with zero hyperbolic step.*

The Jacobson Radical of the Semicrossed Product

Corollary

$\mathcal{A}(\mathbb{D}) \times_{\alpha} \mathbb{Z}^{+}$ has nonzero quasinilpotent elements if φ is hyperbolic or parabolic with positive hyperbolic step.

Corollary

$\mathcal{A}(\mathbb{D}) \times_{\alpha} \mathbb{Z}^{+}$ is semi-simple if and only if the recurrent points are dense in $\mathbb{T}$.

Elliptic case

Lemma

If φ is elliptic with the Denjoy-Wolff point 0, then φ is measure-preserving.

Lemma

For any nonzero $f \in \mathcal{A}(\mathbb{D})$, if φ is elliptic with the Denjoy-Wolff point 0, then Uf is not a quasinilpotent element of $\mathcal{A}(\mathbb{D}) \times_\alpha \mathbb{Z}^+$.

Proof.

- Note that $||(Uf)^n||^{\frac{1}{n}} = \sup_{x\in\mathbb{T}} |f(x)f\circ\varphi(x)\dots f\circ\varphi^{n-1}(x)|^{\frac{1}{n}}$.
- $f \in \mathcal{A}(\mathbb{D}) \rightarrow \log|f|$ is integrable.
- Since φ is measure-preserving and the Ergodic Theorem,

$$\log|f(x)f\circ\varphi(x)\dots f\circ\varphi^{n-1}(x)|^{\frac{1}{n}}$$
$$= \frac{1}{n}\sum_{k=0}^{n-1}\log|f\circ\varphi^k(x)| \rightarrow \int_{\mathbb{T}}\log|f|dm \text{ a.e.}$$

Proof (continue).

- Let $x_0 \in \mathbb{T}$ be such that the above convergence holds. Then

$$\begin{aligned}
&\lim_{n\to\infty} ||(Uf)^n||^{\frac{1}{n}} \\
&\geq \lim_{n\to\infty} |f(x_0)f\circ\varphi(x_0)\ldots f\circ\varphi^{n-1}(x_0)|^{\frac{1}{n}} \\
&= \exp\{\lim_{n\to\infty} \log|f(x_0)f\circ\varphi(x_0)\ldots f\circ\varphi^{n-1}(x_0)|^{\frac{1}{n}}\} \\
&= \exp\{\int_{\mathbb{T}} \log|f|dm\} \\
&> 0.
\end{aligned}$$

□

Isomorphism of Semicrossed Products

Theorem

Let φ and ψ be non-trivial finite Blaschke products. If φ and ψ are conjugate, $\psi = \tau^{-1} \circ \varphi \circ \tau$ for some conformal mapping τ, then $\mathbb{Z}^+ \times_\alpha \mathcal{A}(\mathbb{D})$ is isomorphic to $\mathbb{Z}^+ \times_\beta \mathcal{A}(\mathbb{D})$, where $\alpha(f) = f \circ \varphi$ and $\beta(f) = f \circ \psi, f \in \mathcal{A}(\mathbb{D})$.

Theorem

If φ is elliptic, then $\mathcal{A}(\mathbb{D}) \times_\alpha \mathbb{Z}^+$ has no nonzero quasinilpotent elements.

Proof.

- Any elliptic map is conjugate to an elliptic map fixes zero.
- Let F be a quasinilpotent element, and let $\pi_i(F) = f_i$.
- Then $f_0 = 0$ and suppose that $\exists k > 0$, $f_k \neq 0$ where $f_i = 0 \ \forall i < k$.

Proof (continue).

- Then

$$\begin{aligned} ||F^n||^{\frac{1}{n}} &\geq ||\pi_{nk}(F^n)||^{\frac{1}{n}} \\ &= \sup_{x\in\mathbb{T}} |f_k(x) f_k \circ \varphi^k(x) \dots f_k \circ \varphi^{(n-1)k}(x)|^{\frac{1}{n}} \\ &= ||(U_\beta f_k)^n||^{\frac{1}{n}}, \end{aligned}$$

  where $U_\beta f_k \in A(\mathbb{D}) \times_\beta \mathbb{Z}^+$ and $\beta(f) = f \circ \varphi^k$.
- Note φ^k is also elliptic the Denjoy-Wolff point 0.
- By previous lemma, $\lim_{n\to\infty} ||F^n||^{\frac{1}{n}} \geq \lim_{n\to\infty} ||(U_\beta f_k)^n||^{\frac{1}{n}} > 0$.
- F is not quasinilpotent, a contradiction.
- Hence, $F = 0$.

□

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Thank you!!!