Extensions of Hilbert Modules over Tensor Algebras

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Outline of topics

C^*-correspondences

- A - a unital C^* algebra
- X a C^*-correspondence. Recall that this means X is a certain kind of bimodule over A. Specifically,
  - X is a right Hilbert C^*-module over A.
  - Its left A-action is given by a C^*-homomorphism $\phi : A \to \mathscr{L}(X)$.

Tensor Powers

$X^{\otimes 2} = X \otimes_A X$ is a C^*-correspondence satisfying

- $a \cdot (x \otimes y) := \phi(a)x \otimes y$.
- $(x \otimes y) \cdot b := x \otimes yb$.
- $xa \otimes y := x \otimes \phi(a)y$.
- $\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle := \langle x_2, \phi(\langle x_1, y_1 \rangle) y_2 \rangle$.

Similarly, define $X^{\otimes 3}, X^{\otimes 4}, \ldots$

Constructing the Tensor Algebra

Form the Fock space:

$$\mathscr{F}(X) := A \oplus X \oplus X^{\otimes 2} \oplus X^{\otimes 3} \oplus \cdots .$$

Define $\phi_\infty : A \to \mathscr{L}(\mathscr{F}(X))$ by

$$\phi_\infty(a) = \begin{bmatrix} a & & & & \\ & \phi(a) & & & \\ & & \phi_2(a) & & \\ & & & \phi_3(a) & \\ & & & & \ddots \end{bmatrix}$$

where $\phi_n(a)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = (\phi(a)x_1) \otimes x_2 \otimes \cdots \otimes x_n$.

Constructing the Tensor Algebra

Fock space $= \mathscr{F}(X) := A \oplus X \oplus X^{\otimes 2} \oplus X^{\otimes 3} \oplus \cdots$

For each $x \in X$, we define the creation operator $T_x \in \mathscr{L}(\mathscr{F}(X))$ by

$$T_x = \begin{bmatrix} 0 & & & & \\ T_x^{(1)} & 0 & & & \\ & T_x^{(2)} & 0 & & \\ & & T_x^{(3)} & 0 & \\ & & & \ddots & \ddots \end{bmatrix}$$

where $T_x^{(k)} : X^{\otimes k} \to X^{\otimes (k+1)}$ is $T_x^{(k)}(x_1 \otimes \cdots \otimes x_k) = x \otimes x_1 \otimes \cdots \otimes x_k$.

Constructing the Tensor Algebra

Definition

The *tensor algebra* of X, denoted $\mathscr{T}_+(X)$, is the norm closed subalgebra of $\mathscr{L}(\mathscr{F}(X))$ generated by $\phi_\infty(A)$ and $\{T_x | x \in X\}$.

Examples

1. $A = X = \mathbb{C}, \mathscr{T}_+(X) = A(\mathbb{D})$ - *classical disc algebra*
2. $A = \mathbb{C}, X = \mathbb{C}^d, \mathscr{T}_+(X) = \mathscr{A}_d$ - Popescu's *noncommutative disc algebra*
3. Let α be an automorphism of a unital C^*-algebra A. Let $X = {}_\alpha A$ by defining
   1. $x \cdot a := xa$.
   2. $\phi(a)x = \alpha(a)x$.
   3. $\langle x, y \rangle := x^* y$.

   - $\phi : A \to \mathscr{L}(A)$ equals α since $\mathscr{L}(A) = M(A) = A$.
   - $\mathscr{F}(X) = \ell^2(\mathbb{Z}^+; A)$
   - $\mathscr{T}_+(X)$ is generated by $\phi_\infty(A)$ and $S = T_1$, a shift.
   - $\mathscr{T}_+(X) = A \times_\alpha \mathbb{Z}^+$ is the analytic crossed product of A by $\mathbb{Z}^+$ determined by α.

Modules

Definition

1. A Hilbert space H is a *(c.b.) Hilbert module* over an operator algebra B if the action of B on H is given by a completely bounded homomorphism $\pi : B \to B(H)$.
2. $\varphi : H \to H'$ is a *Hilbert module map* if it is a B-module map between Hilbert modules that is bounded as a Hilbert space operator.

Note: We will assume $A \subset B$ is a C^*-algebra, although B need not be self-adjoint. Furthermore, the representation $(\pi|_A) : A \to B(H)$ is a C^*-representation.

Extensions

Definition

An extension ξ is a short exact sequence

$$\xi : 0 \longrightarrow H \xrightarrow{\varphi} J \xrightarrow{\psi} K \longrightarrow 0$$

where H, J, and K are Hilbert modules over an operator algebra B and φ and ψ are Hilbert-module maps.

Note: In particular, the range of φ equals the kernel of ψ. So φ is bounded below and ψ is bounded below on its initial space.

Equivalence of Extensions

Two extensions ξ and ξ' are equivalent if and only if there exist a Hilbert-module map $\theta : J \to J'$ making the following diagram commute:

$$\begin{array}{ccccccccc}
\xi : 0 & \longrightarrow & H & \xrightarrow{\varphi} & J & \xrightarrow{\psi} & K & \longrightarrow & 0 \\
 & & \| & & \downarrow \theta & & \| & & \\
\xi' : 0 & \longrightarrow & H' & \xrightarrow{\varphi'} & J' & \xrightarrow{\psi'} & K' & \longrightarrow & 0
\end{array}$$

The collection (in fact group) of equivalence classes of extensions is denoted $\mathrm{Ext}^1(K, H)$.

Hilbert Space Decomposition

$$\xi : 0 \longrightarrow H \xrightarrow{\varphi} J \xrightarrow{\psi} K \longrightarrow 0$$

As Hilbert spaces, $J \cong H \oplus K$ (but not neccesarily as B-modules.)

Cocycles

$$\xi : 0 \longrightarrow H \xrightarrow{\varphi} H \oplus K \xrightarrow{\psi} K \longrightarrow 0$$

Let $\pi : B \to B(H)$ and $\rho : B \to B(K)$ be the representations of B on H and K, respectively.

Derivations

The B-module action on $H \oplus K$, is given by

$$\begin{pmatrix} \pi(\cdot) & \delta(\cdot) \\ 0 & \rho(\cdot) \end{pmatrix} : B \to B(H \oplus K)$$

where $\delta : B \to B(K, H)$ is a completely bounded A-derivation

1. $\delta(fg) = \delta(f)\rho(g)k + \pi(f)\delta(g) \quad \forall f, g \in B$
2. $\delta(a) = 0$ for all $a \in A$.

Note: δ is, technically, a $\phi_\infty(A)$-derivation).

Equivalence of Extensions

If the derivations δ and δ' correspond, respectively, to extensions ξ and ξ', then $\xi \approx \xi'$ if and only if $\delta - \delta'$ is an *inner* derivation: there exists $L \in B(K, H)$ such that

$$(\delta - \delta')(f) = \pi(f)L - L\rho(f) \forall f \in B.$$

An inner derivation is A-linear iff $\pi(a)L = L\rho(a) \quad \forall a \in A.$

Cocycles

Alternatively, we can describe extensions in terms of *cocycles*:

Definition

A *cocycle* is a bilinear map $\sigma : B \times K \to H$ satisfying

$$\sigma(fg, k) = \pi(f)\sigma(g, k) + \sigma(f, \rho(g)k).$$

which is completely bounded when H and K are given their column Hilbert space structure.

Derivations and cocycles are related via the equation

$$\sigma(f, k) = \delta(f)k.$$

Extension Equivalence

$\xi \approx \xi'$ if and only if

$$\sigma(f,k) - \sigma'(f,k) = \pi(f)Lk - L\rho(f)k.$$

Product Rule

Proposition

Suppose H and K are Hilbert modules over B with representations $\pi : \mathcal{T}_+({}_\alpha A) \to B(H)$ and $\rho : \mathcal{T}_+({}_\alpha A) \to B(K)$, respectively. If $\sigma : \mathcal{T}_+({}_\alpha A) \times K \to H$ is a cocycle, then

$$\sigma(S^{n+1}, k) = \sum_{j=0}^{n} \pi(S^{n-j})\sigma(S, \rho(S^j)k)$$

for every $n \geq 0, S \in B, k \in K$.

Induced Representation

- Let $\psi : A \to B(E)$ be a representation and let $\{e_m\}_{m\geq 0}$ be an orthonormal basis for E.
- From now on, we only consider $B = \mathscr{T}_+({}_\alpha A)$ and $H = \ell^2(\mathbb{Z}^+; A) \otimes_\psi E$.
- $\{\delta_n \otimes e_m\}_{n,m\geq 0}$ is an orthonormal basis for $\ell^2(\mathbb{Z}^+; A) \otimes_\psi E$., where $\delta_n(k) = \delta_{nk} 1_A$.
- $\pi : \mathscr{T}_+({}_\alpha A) \to B(\ell^2(\mathbb{Z}^+; A) \otimes_\psi E)$ is given by $\pi|_A = \phi_\infty \otimes id_E$ and $\pi(T_1) = U_+ \otimes id_E$.

Cocycles Defined by Vectors

Definition

We say a sequence of vectors in K, $\{\mathbf{k_m}\}$ **define a cocycle** σ if $\sigma(S, k) = \sum_m \langle k, k_m \rangle \delta_0 \otimes e_m$.

Motivation

Theorem (Carlson & Clark, 1995)

Let K be a Hilbert $A(\mathbb{D})$-module. Then a vector $k_0 \in K$ defines a cocycle $\sigma : A(\mathbb{D}) \times K \to H^2$ if and only if

$$\sum_{n=0}^{\infty} |\langle \rho(S^n)k, k_0 \rangle|^2 < \infty$$

for all $k \in K$.

Note: H^2 is the classical Hardy space and $\sigma(S, k) = \langle k, k_0 \rangle \in H^2$.

Boundedness Criterion

Theorem (Greene, 2011)

Let K be a Hilbert $\mathcal{T}_+({}_\alpha A)$-module. Then a sequence in K, $\{k_m\}_{m=0}^\infty$ defines a cocycle $\sigma : \mathscr{T}_+({}_\alpha A) \times K \to \ell^2(\mathbb{Z}^+; A) \otimes_\psi E$ if and only if

1. $$\sum_{n=0}^\infty \sum_{m=0}^\infty |\langle \rho(S^n)k, k_m\rangle|^2 < \infty \quad \forall k \in K$$

2. $$\pi(\alpha(a))k_m = \sum_{m'} \langle \psi(a)e_m, e_{m'}\rangle k_{m'}$$

Corollary

Corollary

If $N = \dim(E) < \infty$ *and* $\mathrm{sp}(\rho(S)) \subset \mathbb{D}$*, then any* $\{k_m\}_{1 \leq m \leq N}$ *satisfying* (2) *defines a cocycle* σ*.*

Proof.

Define the functions $h_m(z) = \langle \sum_n (z\rho(S))^n k, k_m \rangle$. By hypothesis $h_m(z) = \left\langle (id_K - z\rho(S))^{-1} k, k_m \right\rangle$ for $|z| < \|\rho(S)\|^{-1}$ and $h_m(z)$ are analytic across the unit circle. □

Proof Continued

Continuation of proof.

$$\begin{aligned}\sum_{m=1}^{N}\sum_{n=0}^{\infty}|\langle z\rho(S^n)k, k_m\rangle|^2 &= \|\sum_{n,m}\langle z\rho(S)^n k, k_m\rangle\delta_n\otimes e_m\| \\ &\leq \sum_{m}\|\langle(id_K - z\rho(S))^{-1}k, k_m\rangle\| \\ &\leq \sum_{m=1}^{N}\|h_m(z)\| \\ &< \infty.\end{aligned}$$

□

Corollary

Corollary

If $\rho(S) = id_K$, then $\{k_m\}$ defines a cocycle σ only if $k_m = 0$ for every m. It follows that $\mathrm{Ext}(K, \ell^2(\mathbb{Z}^+; A) \otimes_\psi E) = 0$.

Proof.

$$\sum_{n,m} |\langle \rho(S^n)k, k_m\rangle|^2 = \sum_{n,m} |\langle k, k_m\rangle|^2 < \infty \iff k_m = 0 \forall m.$$

□

Characterization of Cocycles

Theorem (Greene, 2011)

Every cocycle σ is equivalent to a cocycle defined by some $\{k_m\}$.

Proof.

1. Let σ be a cocycle.
2. By the Riesz Representation theorem, there exist $K_{n,m} \in K$ with
$$\sigma(S, k) = \sum_{n,m} \langle k, K_{n,m} \rangle \delta_n \otimes e_m.$$

□

Characterization of Cocycles

Proof.

3 By the product formula,

$$\begin{aligned}
\sigma(S^{N+1}, k) &= \sum_{j=0}^{N} \pi(S^{N-j})\sigma(S, \rho(S^j)k) \\
&= \sum_{j=0}^{N} \pi(S^{N-j}) \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \langle \rho(S^j)k, K_{n,m}\rangle \delta_n \otimes e_m \\
&= \sum_{j=0}^{N}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \langle k, \rho(S)^{*j} K_{n,m}\rangle \delta_{N+n-j} \otimes e_m \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{N}\sum_{m=0}^{\infty} \langle k, \rho(S)^{*j} K_{n,m}\rangle \delta_{N+n-j} \otimes e_m
\end{aligned}$$

□

Characterization of Cocycles

Proof.

4 The coefficient of the $\delta_\nu \otimes e_m$ term of $\sigma(S^{N+1}, k)$ is

$$\begin{cases} \sum_{j=0}^{N} \langle k, \rho(S)^{*j} K_{\nu+j-N,m} \rangle & \text{for } \nu \geq N \\ \sum_{j=0}^{\nu} \langle k, \rho(S)^{*N-\nu+j} K_{j,m} \rangle & \text{for } \nu < N. \end{cases}$$

5 Therefore, $\left\{ \left\langle k, \sum_{j=1}^{N} \rho(S)^{*j} K_{j+p,m} \right\rangle \right\}_{N=1}^{\infty}$ is a bounded sequence in N.

6 Letting Lim be a Banach limit on ℓ^∞, we define $k_{p,m} \in K$ by

$$\langle k, k_{p,m} \rangle = \mathrm{Lim}_{N \to \infty} \left\langle k, \sum_{j=0}^{N} \rho(S)^{*j} K_{j+p,m} \right\rangle .$$

□

Characterization of Cocycles

Proof.

7 Define σ_0 by $\sigma_0(S,k) = \sum_m \langle k, k_{0,m}\rangle \delta_0 \otimes e_m$.
Note: σ_0 is A-linear iff $\pi(\alpha(a))k_{0,m} = \sum_p \langle \psi(a)e_m, e_p\rangle k_{0,p}$.

8 Define $L : K \to \ell^2(\mathbb{Z}^+; A) \otimes_\psi E$ by
$Lk = \sum_{j,m} \langle k, k_{j+1,m}\rangle \delta_j \otimes e_m$.

9 $\sigma(S,k) - \sigma_0(S,k) = (\pi(S)L - L\rho(S))k$.

□

Ongoing and Future Work

1. Characterize the coboundaries.
2. Calculate $\mathrm{Ext}^1(K, \ell^2(\mathbb{Z}^+; A) \otimes_\psi E)$.
3. Study the more general setting with $\alpha \in End(A)$.
4. Generalize to $\mathscr{T}_+(X)$.
5. Study projectivity and injectivity in terms of Ext.

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