Tightness and Inverse Semigroups

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April 14, 2012

This work is a) joint with David Milan, and b) in progress.

Combinatorial Object	$\rightarrow$	C^*-algebra
$\downarrow$		$\uparrow$
Inverse Semigroup	$\rightarrow$	Groupoid

- Inverse Semigroups
- Tight Representations and C^*-algebras
- An alternative
- Speculations about Relative Categories of Paths

An *inverse semigroup* is a semigroup so that for every s there is a unique element, called s^*, so that

$$ss^*s = s, \qquad s^*ss^* = s^*.$$

They can also be characterized a (von Neumann) regular semigroups in which the idempotents commute. Our inverse semigroups are discrete.

Use $E(S)$ for the idempotents of S.

One example is the partial 1-1 maps on a set X, $I(X)$, called the *symmetric inverse monoid*, in analogy with the symmetric group of bijections on X.

A natural example (at INFAS) is a collection of partial isometries on Hilbert space with commuting initial and final projections. In fact, Duncan and Paterson showed that every inverse semigroup can be represented this way on Hilbert space.
For combinatorial objects, we have associated inverse semigroups. For example, for a graph G, we have an inverse semigroup $S(G)$ whose elements are 0 and pq^* where p and q are paths in the graph with the same endpoint.

For any inverse semigroup S, there is a universal C^*-algebra, $C^*(S)$, with the usual universal property that each representation of S lifts to a unique representation of $C^*(S)$.
But $C^*(S(G))$ is **not** the C^*-algebra of the graph G. It is the Toeplitz C^*-algebra, so $C^*(G)$ is a quotient of $C^*(S(G))$.
Ruy Exel has a fix for this, by looking a smaller C^*-algebra associated to $S(G)$, the *tight* C^*-algebra of $S(G)$.

The natural partial order on an inverse semigroup. For two elements a and b in an inverse semigroup S, we say

$$a \leq b \quad \Leftrightarrow \quad a = be$$

for some idempotent e. (Think restriction in $I(X)$.)
If $a \leq b$, then ab^{-1} and $a^{-1}b$ are idempotents (but not conversely).
If $e, f \in E(S)$, then ef is a meet for e and f, but two elements of S need not have a meet.
For idempotents e and f, we write $e \Cap f$ if there is a nonzero idempotent g with $g \leq e$ and $g \leq f$; otherwise, write $e \perp f$.

Pick finite sets of idempotents X and Y and define

$$E^{X,Y} := \{z \in E(S) : z \leq x \forall x \in X, z \perp y \forall y \in Y\}.$$

A cover of $E^{X,Y}$ is a finite set $C \subseteq E^{X,Y}$ so that for all nonzero $a \in E^{X,Y}$, there is $c \in C$ with $c \Cap a$.
Suppose $\beta : E(S) \to B$ is a representation of the meet-semilattice $E(S)$ in a Boolean algebra B. Exel defined β to be *tight* if, for all finite sets X and Y in $E(S)$ and all covers C of $E^{X,Y}$,

$$\vee\beta(C) = (\wedge\beta(X)) \bigwedge (\neg \vee \beta(Y)).$$

A representation $\gamma : S \to T$, where $E(T)$ is Boolean, is *tight* if $\gamma|_{E(S)}$ is tight as above.

Theorem (Exel)

For a countable inverse semigroup S with zero, there is a C^*-algebra, $C_t^*(S)$ so that there is a one-to-one correspondence between tight Hilbert space representations of S and $*$-representations of $C_t^*(S)$.

Moreover, $C_t^*(S(G)) = C^*(G)$, and similar statements hold for C*-algebras of semigroupoids and tilings, for example.
But . . . .

Since inverse semigroups have a natural partial order, we can talk about joins of elements.
If a and b have a join, c, then $a, b \leq c$ and this forces

$$ab^{-1}, a^{-1}b \in E(S).$$

Call a and b *compatible* if this holds.
An inverse semigroup is *complete* if, for all compatible finite sets, there is a join. It is *distributive* if, for all elements a and finite sets B,

$$a\left(\bigvee B\right) = \bigvee aB.$$

The idea of tightness is: if Z is a cover for $E^{X,Y}$ then the join of Z should equal the meet of X minus the meet of Y.
So lets build these joins into the inverse semigroup.

Following Lenz, for an element a and a finite set B, we write $a \to B$ if for all $y \leq a$, there is $b \in B$ with $b \Cap y$.
Reserving E-cover for the old notion of cover, call a set C a *cover* of an element a if $c \leq a$ for all $c \in C$ and $a \to C$.
A map $\gamma : S \to T$ into an complete, distributive inverse semigroup T is a *cover-to-join* map if for each $a \in S$ and each cover of a, C,

$$\gamma(s) = \bigvee \theta(C).$$

Proposition

Let S be an 'nice' inverse semigroup with zero and T a complete distributive inverse semigroup with $E(T)$ a Boolean algebra.
Then a homomorphism $\theta : S \to T$ is tight if and only if it is join-to-cover.

The key points of the proof are

- if Z is a cover for x, then Z is an E-cover for $E^{\{x\},\varnothing}$,
- if Z is an E-cover for $E^{X,Y}$, then $Y \cup Z$ is a cover for $\wedge X$.

Note that if $\beta : S \to B(\mathcal{H})$ then we can regard $\beta(S) \subset B(\mathcal{H})$ as a complete, distributive inverse semigroup.
There are tight maps that are not cover-to-join, but their co-domains are not complete or distributive.

'nice' in the previous slide involves a technical condition analoguous to finitely aligned in higher rank graphs.
For a compatible set A in an inverse semigroup, use $A^{\downarrow}$ for all elements below some element of A.
'nice' means For all x and y, there is a finite set A, possibly 0, so that

$$x^{\downarrow} \cap y^{\downarrow} = A^{\downarrow}.$$

Theorem (Lawson, adapted)

For a 'nice' inverse semigroup S, there is a distributive, complete inverse semigroup $D(S)$ and a cover-to-join h'ism $\delta : S \to D(S)$ so that, for every join-to-cover map $\theta : S \to T$, with T a complete distributive inverse semigroup, there is a unique join-preserving homomorphism $\overline{\theta} : D(S) \to T$ so that $\theta = \overline{\theta}\delta$.

Let $C_j^*(T)$ be the C*-algebra universal for the family of join-preserving Hilbert space representations of T.

Corollary

$$C_j^*(D(S)) \cong C_t^*(S).$$