Commutative algebras of Toeplitz operators in action

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Toeplitz operators

The Toeplitz operator was originally defined in terms of the so-called Toeplitz matrix

$$A = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

where $a_n \in \mathbb{C}$, $n \in \mathbb{Z}$.

Theorem (O.Toeplitz, 1911)

Matrix A defines a bounded operator on $l_2 = l_2(\mathbb{Z}_+)$, where $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$, if and only if the numbers $\{a_n\}$ are the Fourier coefficients of a function $a \in L_\infty(S^1)$, where S^1 is the unit circle.

Hardy space version

The (discrete) Fourier transform $\mathcal{F}$ is a unitary operator which maps $L_2(S^1)$ onto $l_2(\mathbb{Z})$ and the Hardy space $H^2_+(S^1)$ onto $l_2(\mathbb{Z}_+)$. Then for the operator $\mathbb{A}$, defined by the matrix A we have

$$\mathcal{F}^{-1}\,\mathbb{A}\,\mathcal{F} = T_a : H^2_+(S^1) \longrightarrow H^2_+(S^1).$$

The operator T_a acts on the Hardy space $H^2_+(S^1)$ by the rule

$$T_a :\ f(t) \in H^2_+(S^1) \longmapsto (P_+ af)(t) \in H^2_+(S^1),$$

where $P_+ : L_2(S^1) \longrightarrow H^2_+(S^1)$ is the Szegö orthogonal projection, and the Fourier coefficients of the function a are given by the sequence $\{a_n\}$.

Operator theory version

Let H be a Hilbert space, H_0 be its subspace.
Let $P_0 : H \longmapsto H_0$ be the orthogonal projection,
and let A be a bounded linear operator on H.

The Toeplitz operator with symbol A

$$T_A : x \in H_0 \longmapsto P_0(Ax) \in H_0$$

is the compression of A (in our case of a multiplication operator) onto the subspace H_0, representing thus an important model case in operator theory.

Bergman space version

Consider now $L_2(\mathbb{D})$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$.
The Bergman space $\mathcal{A}^2(\mathbb{D})$ is the subspace of $L_2(\mathbb{D})$ consisting of functions analytic in $\mathbb{D}$.
The Bergman orthogonal projection $B_{\mathbb{D}}$ of $L_2(\mathbb{D})$ onto $\mathcal{A}^2(\mathbb{D})$ has the form

$$(B_{\mathbb{D}}\varphi)(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{\varphi(\zeta)\, d\mu(\zeta)}{(1 - z\overline{\zeta})^2},$$

The Toeplitz operator T_a with symbol $a = a(z)$ acts as follows

$$T_a : \varphi(z) \in \mathcal{A}^2(\mathbb{D}) \longmapsto (B_{\mathbb{D}}\, a\varphi)(z) \in \mathcal{A}^2(\mathbb{D}).$$

Metric

Consider the unit disk $\mathbb{D}$ endowed with the hyperbolic metric

$$g = ds^2 = \frac{1}{\pi} \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}.$$

A geodesic in $\mathbb{D}$ is (a part of) an Euclidean circle or a straight line orthogonal to the boundary $S^1 = \partial\mathbb{D}$.

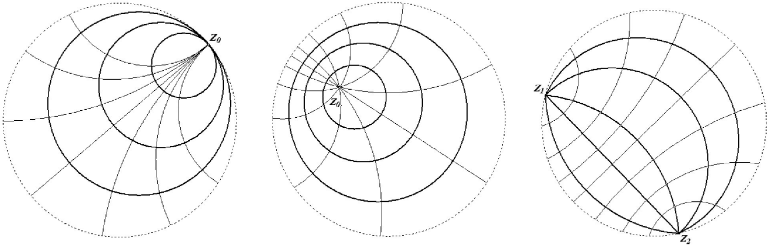
Each pair of geodesics, say L_1 and L_2, lie in a geometrically defined object, one-parameter family $\mathcal{P}$ of geodesics, which is called the pencil determined by L_1 and L_2.

Each pencil has an associated family $\mathcal{C}$ of lines, called cycles, the orthogonal trajectories to geodesics forming the pencil.

Pencils of hyperbolic geodesics

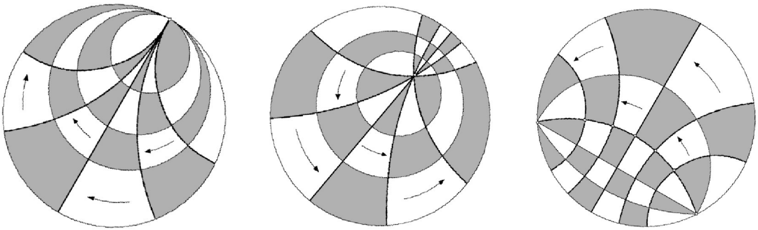
There are three types of pencils of hyperbolic geodesics:

- parabolic,
- elliptic,
- hyperbolic.



Möbius transformations

Each Möbius transformation $g \in \mathrm{M\ddot{o}b}(\mathbb{D})$ is a movement of the hyperbolic plane, determines a certain pencil of geodesics $\mathcal{P}$, and its action is as follows:
each geodesic L from the pencil $\mathcal{P}$, determined by g, moves along the cycles in $\mathcal{C}$ to the geodesic $g(L) \in \mathcal{P}$, while each cycle in $\mathcal{C}$ is invariant under the action of g



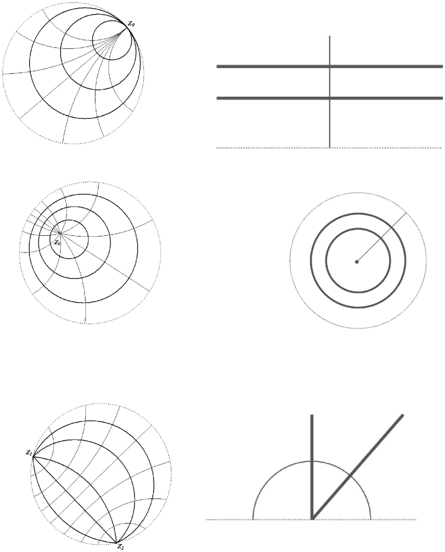
Miracle

Theorem

Given a pencil $\mathcal{P}$ of geodesics, consider the set of symbols which are constant on corresponding cycles. The C^*-algebra generated by Toeplitz operators with such symbols is commutative.

That is, each pencil of geodesics generates a commutative C^*-algebra of Toeplitz operators.

Model cases



Hyperbolic case

Consider the upper half-plane Π, the space $L_2(\Pi)$, and its Bergman subspace $\mathcal{A}^2(\Pi)$. We construct the operator

$$R: \; L_2(\Pi) \longrightarrow L_2(\mathbb{R}),$$

whose restriction onto the Bergman space

$$R|_{\mathcal{A}^2(\Pi)} : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R})$$

is an isometric isomorphism.
The ajoint operator

$$R^* : \; L_2(\mathbb{R}) \longrightarrow \mathcal{A}^2(\Pi) \subset L_2(\Pi)$$

is an isometric isomorphism of $L_2(\mathbb{R})$ onto $\mathcal{A}^2(\Pi)$.
Moreover we have

$$\begin{aligned} R\,R^* = I \quad &: \quad L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}), \\ R^* R = B_\Pi \quad &: \quad L_2(\Pi) \longrightarrow \mathcal{A}^2(\Pi). \end{aligned}$$

Hyperbolic case

Theorem

Let $a = a(\theta) \in L_\infty(\Pi)$ be a homogeneous of order zero function, (a functions depending only on the polar angle θ).
Then the Toeplitz operator T_a acting on $\mathcal{A}^2(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_a I = R\,T_a R^*$, acting on $L_2(\mathbb{R})$.
The function $\gamma_a(\lambda)$ is given by

$$\gamma_a(\lambda) = \frac{2\lambda}{1 - e^{-2\pi\lambda}} \int_0^\pi a(\theta)\, e^{-2\lambda\theta}\, d\theta, \qquad \lambda \in \mathbb{R}.$$

Symplectic manifold

We consider the pair $(\mathbb{D}, \omega)$, where $\mathbb{D}$ is the unit disk and

$$\omega = \frac{1}{\pi} \frac{dx \wedge dy}{(1-(x^2+y^2)^2} = \frac{1}{2\pi i} \frac{d\overline{z} \wedge dz}{(1-|z|^2)^2}.$$

Poisson brackets:

$$\begin{aligned} \{a, b\} &= \pi(1-(x^2+y^2))^2 \left(\frac{\partial a}{\partial y}\frac{\partial b}{\partial x} - \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} \right) \\ &= 2\pi i (1-z\overline{z})^2 \left(\frac{\partial a}{\partial z}\frac{\partial b}{\partial \overline{z}} - \frac{\partial a}{\partial \overline{z}}\frac{\partial b}{\partial z} \right). \end{aligned}$$

Laplace-Beltrami operator:

$$\begin{aligned} \Delta &= \pi(1-(x^2+y^2))^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \\ &= 4\pi(1-z\overline{z})^2 \frac{\partial^2}{\partial z \partial \overline{z}}. \end{aligned}$$

Introduce weighted Bergman spaces $\mathcal{A}_h^2(\mathbb{D})$ with the scalar product

$$(\varphi, \psi) = \left(\frac{1}{h} - 1\right) \int_{\mathbb{D}} \varphi(z)\overline{\psi(z)}\,(1 - z\overline{z})^{\frac{1}{h}}\,\omega(z).$$

The weighted Bergman projection has the form

$$(B_{\mathbb{D},h}\varphi)(z) = \left(\frac{1}{h} - 1\right) \int_{\mathbb{D}} \varphi(\zeta)\, \left(\frac{1 - \zeta\overline{\zeta}}{1 - z\overline{\zeta}}\right)^{\frac{1}{h}} \omega(\zeta).$$

Let $E = (0, \frac{1}{2\pi})$, for each $\hbar = \frac{h}{2\pi} \in E$, and consequently $h \in (0, 1)$, introduce the Hilbert space $H_\hbar$ as the weighted Bergman space $\mathcal{A}_h^2(\mathbb{D})$.

Wick symbol

For each function $a = a(z) \in C^\infty(\mathbb{D})$ consider the family of Toeplitz operators $T_a^{(h)}$ with (anti-Wick) symbol a acting on $\mathcal{A}_h^2(\mathbb{D})$, for $h \in (0,1)$, and denote by $\mathcal{T}_h$ the *-algebra generated by Toeplitz operators $T_a^{(h)}$ with symbols $a \in C^\infty(\mathbb{D})$.

The Wick symbols of the Toeplitz operator $T_a^{(h)}$ has the form

$$\widetilde{a}_h(z,\overline{z}) = (\frac{1}{h} - 1) \int_{\mathbb{D}} a(\zeta) \left(\frac{(1-|z|^2)(1-|\zeta|^2)}{(1-z\overline{\zeta})(1-\zeta\overline{z})} \right)^{\frac{1}{h}} \omega(\zeta).$$

Star product

For each $h \in (0,1)$ define the function algebra

$$\widetilde{\mathcal{A}}_h = \{\widetilde{a}_h(z,\overline{z}) : \ a \in C^\infty(\mathbb{D})\}$$

with point wise linear operations, and with the multiplication law defined by the product of Toeplitz operators:

$$\widetilde{a}_h \star \widetilde{b}_h = (\frac{1}{h} - 1) \int_{\mathbb{D}} \widetilde{a}_h(z,\overline{\zeta})\, \widetilde{b}_h(\zeta,\overline{z}) \left(\frac{(1-|z|^2)(1-|\zeta|^2)}{(1-z\overline{\zeta})(1-\zeta\overline{z})} \right)^{\frac{1}{h}} \omega.$$

Correspondence principle

The correspondence principle is given by

$$\begin{aligned} \widetilde{a}_h(z,\overline{z}) &= a(z,\overline{z}) + O(\hbar), \\ (\widetilde{a}_h \star \widetilde{b}_h - \widetilde{b}_h \star \widetilde{a}_h)(z,\overline{z}) &= i\hbar\,\{a,b\} + O(\hbar^2). \end{aligned}$$

Three term asymptotic expansion

$$
\begin{aligned}
&(\widetilde{a}_h \star \widetilde{b}_h - \widetilde{b}_h \star \widetilde{a}_h)(z,\overline{z}) = \\
&i\hbar \{a, b\} + \\
&i\frac{\hbar^2}{4} (\Delta\{a, b\} + \{a, \Delta b\} + \{\Delta a, b\} + 8\pi\{a, b\}) + \\
&i\frac{\hbar^3}{24} \big[\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} + \Delta^2\{a, b\} + \\
&\qquad \Delta\{a, \Delta b\} + \Delta\{\Delta a, b\} + \\
&\qquad 28\pi (\Delta\{a, b\} + \{a, \Delta b\} + \{\Delta a, b\}) + 96\pi^2\{a, b\}\big] + \\
&o(\hbar^3)
\end{aligned}
$$

Corollary

Let $\mathcal{A}(\mathbb{D})$ be a subspace of $C^\infty(\mathbb{D})$ such that for each $h \in (0,1)$ the Toeplitz operator algebra $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ is commutative. Then for all $a,\ b \in \mathcal{A}(\mathbb{D})$ we have

$$\begin{aligned} \{a, b\} &= 0, \\ \{a, \Delta b\} + \{\Delta a, b\} &= 0, \\ \{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} &= 0. \end{aligned}$$

Let $\mathcal{A}(\mathbb{D})$ be a linear space of smooth functions which generates the commutative C^*-algebra $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ of Toeplitz operators for each $h \in (0,1)$.

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First term: $\{a, b\} = 0$:

Lemma

All functions in $\mathcal{A}(\mathbb{D})$ have (globally) the same set of level lines and the same set of gradient lines.

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Lemma

All functions in $\mathcal{A}(\mathbb{D})$ have (globally) the same set of level lines and the same set of gradient lines.

Second term: $\{a, \Delta b\} + \{\Delta a, b\} = 0$**:**

Theorem

The space $\mathcal{A}(\mathbb{D})$ consists of functions whose common gradient lines are geodesics in the hyperbolic geometry of the unit disk $\mathbb{D}$.

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Theorem

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Third term: $\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} = 0$**:**

Theorem

The space $\mathcal{A}(\mathbb{D})$ consists of functions whose common level lines are cycles.

Main theorem

Theorem

Let $\mathcal{A}(\mathbb{D})$ be a space of smooth functions. Then the following two statements are equivalent:

- *there is a pencil $\mathcal{P}$ of geodesics in $\mathbb{D}$ such that all functions in $\mathcal{A}(\mathbb{D})$ are constant on the cycles of $\mathcal{P}$;*
- *the C^*-algebra generated by Toeplitz operators with $\mathcal{A}(\mathbb{D})$-symbols is commutative on each weighted Bergman space $\mathcal{A}_h^2(\mathbb{D})$, $h \in (0, 1)$.*

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Continuous symbols

Let $\mathcal{T}(C(\overline{\mathbb{D}}))$ be the C^*-algebra generated by T_a, with $a \in C(\overline{\mathbb{D}})$.

Theorem

The algebra $\mathcal{T} = \mathcal{T}(C(\overline{\mathbb{D}}))$ is irreducible and contains the whole ideal $\mathcal{K}$ of compact on $\mathcal{A}^2(\mathbb{D})$ operators. Each operator $T \in \mathcal{T}(C(\overline{\mathbb{D}}))$ is of the form

$$T = T_a + K, \quad \text{where} \quad a \in C(\overline{\mathbb{D}}), \quad K \in \mathcal{K}.$$

The homomorphism

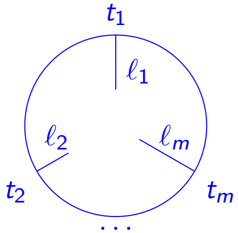
$$\mathrm{sym} : \mathcal{T} \longrightarrow \mathrm{Sym}\, \mathcal{T} = \mathcal{T}/\mathcal{K} \cong C(\partial\mathbb{D})$$

is generated by

$$\mathrm{sym} : T_a \longmapsto a_{|_{\partial\mathbb{D}}}.$$

Piece-wise continuous symbols

Fix a finite number of distinct points $T = \{t_1, ..., t_m\}$ on $\gamma = \partial\mathbb{D}$.
Let ℓ_k, $k = 1, ..., m$, be the part of the radius of $\mathbb{D}$ starting at t_k.
Let $\mathcal{L} = \bigcup_{k=1}^m \ell_k$.



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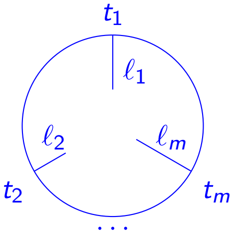

Denote by $PC(\overline{\mathbb{D}}, T)$ the set (algebra) of all piece-wise continuous functions on $\mathbb{D}$ which are

- continuous in $\overline{\mathbb{D}} \setminus \mathcal{L}$,
- have one-sided limit values at each point of $\mathcal{L}$.

Piece-wise continuous symbols

We consider the C^*-algebra $\mathcal{T}_{PC} = \mathcal{T}(PC(\overline{\mathbb{D}}, \ell))$ generated by all Toeplitz operators T_a with symbols $a(z) \in PC(\overline{\mathbb{D}}, \ell)$.

Piece-wise continuous symbols

We consider the C^*-algebra $\mathcal{T}_{PC} = \mathcal{T}(PC(\overline{\mathbb{D}}, \ell))$ generated by all Toeplitz operators T_a with symbols $a(z) \in PC(\overline{\mathbb{D}}, \ell)$.

Bad news: *Let $a(z)$, $b(z) \in PC(\overline{\mathbb{D}}, \ell)$, then*

$$[T_a, T_b) = T_a T_b - T_{ab}$$

is not compact in general.

That is

$$T_a T_b \neq T_{ab} + K.$$

The algebra $\mathcal{T}_{PC}$ has a more complicated structure.

Algebra $\mathcal{T}_{PC}$

For piece-wise continuous symbols the C^*-algebra $\mathcal{T}_{PC}$ contains:

- initial generators T_a, where $a \in PC$,
- $$\sum_{k=1}^{p} \prod_{j=1}^{q_k} T_{a_{j,k}}, \qquad a_{j,k} \in PC,$$
- uniform limits of sequences of such elements.

Compact set Γ

For each $a_1, a_2 \in PC(\overline{\mathbb{D}}, \ell)$ the commutator $[T_{a_1}, T_{a_2}]$ is compact, thus the algebra $\mathrm{Sym}\, \mathcal{T}_{PC}$ is commutative.
And thus

$$\mathrm{Sym}\, \mathcal{T}_{PC} \cong C(\text{over certain compact set } \Gamma).$$

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And thus

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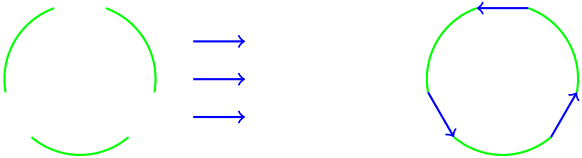
The set Γ is the union $\widehat{\gamma} \cup (\bigcup_{k=1}^{m} [0,1]_k)$, where $\widehat{\gamma}$ be the boundary γ, cut by points $t_k \in T$, with the following point identification

$$t_k - 0 \equiv 0_k, \qquad t_k + 0 \equiv 1_k.$$



Algebra $\mathrm{Sym}\,\mathcal{T}_{PC}$

Theorem

The symbol algebra $\mathrm{Sym}\,\mathcal{T}(PC(\overline{\mathbb{D}},\ell)) = \mathcal{T}(PC(\overline{\mathbb{D}},\ell))/\mathcal{K}$ *is isomorphic and isometric to* $C(\Gamma)$*.*
The homomorphism

$$\mathrm{sym}\ : \mathcal{T}(PC(\overline{\mathbb{D}},\ell)) \to \mathrm{Sym}\,\mathcal{T}(PC(\overline{\mathbb{D}},\ell)) = C(\Gamma)$$

is generated by

$$\mathrm{sym}\ : T_a \longmapsto \begin{cases} a(t), & t \in \widehat{\gamma} \\ a(t_k - 0)(1-x) + a(t_k+0)x, & x \in [0,1] \end{cases},$$

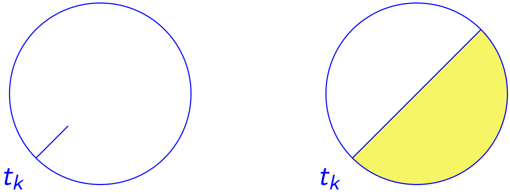
where $t_k \in T$, $k = 1, 2, ..., m$.

Auxiliary functions: χ_k

For each $k = 1, ..., m$, let

$$\chi_k = \chi_k(z)$$

be the characteristic function of the half-disk obtained by cutting $\mathbb{D}$ by the diameter passing through $t_k \in T$, and such that $\chi_k^+(t_k) = 1$, and thus $\chi_k^-(t_k) = 0$.



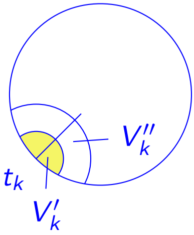
Auxiliary functions: v_k

For two small neighborhoods $V_k' \subset V_k''$ of the point $t_k \in T$, let

$$v_k = v_k(z) : \ \overline{\mathbb{D}} \to [0, 1]$$

be a continuous function such that

$$v_k|_{\overline{V_k'}} \equiv 1, \qquad v_k|_{\overline{\mathbb{D}} \setminus V_k''} \equiv 0.$$



Canonical form of operators: Generators

Let $a \in PC(\overline{\mathbb{D}}, T)$. Then

$$T_a = T_{s_a} + \sum_{k=1}^{m} T_{v_k} p_{a,k}(T_{\chi_k}) T_{v_k} + K,$$

where K is compact, $s_a \in C(\overline{\mathbb{D}})$,

$$\begin{aligned} s_a(z)|_\gamma &\equiv \left[a(z) - \sum_{k=1}^{m} [a^-(t_k) + (a^+(t_k) - a^-(t_k))\chi_k(z)] v_k^2(z) \right]_\gamma, \\ p_{a,k}(x) &= a^-(t_k)(1-x) + a^+(t_k)x. \end{aligned}$$

Canonical form of operators: Sum of products

Let

$$A = \sum_{i=1}^{p} \prod_{j=1}^{q_i} T_{a_{i,j}},$$

then

$$A = T_{s_A} + \sum_{k=1}^{m} T_{v_k} p_{A,k}(T_{\chi_k}) T_{v_k} + K_A,$$

where $s_A = \in C(\overline{\mathbb{D}})$, $p_{A,k} = p_{A,k}(x)$, $k = 1, ..., m$, are polynomials, and K_A is compact.

Canonical form of operators: General operator

Theorem

Every operator $A \in \mathcal{T}(PC(\overline{\mathbb{D}}, T))$ admits the canonical representations

$$A = T_{s_A} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} + K,$$

where $s_A(z) \in C(\overline{\mathbb{D}})$, $f_{A,k}(x) \in C[0,1]$, $k = 1, ..., m$, K is compact.

Toeplitz or not Toeplitz (bounded symbols)

Theorem

An operator

$$A = T_{s_A} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} + K$$

is a compact perturbation of a Toeplitz operator if and only if every operator $f_{A,k}(T_{\chi_k})$ is a Toeplitz operator, where $k = 1, ..., m$.

Theorem

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is a compact perturbation of a Toeplitz operator if and only if every operator $f_{A,k}(T_{\chi_k})$ is a Toeplitz operator, where $k = 1, ..., m$.

Let $f_{A,k}(T_{\chi_k}) = T_{a_k}$ for some $a_k \in L_\infty(\mathbb{D})$. Then $A = T_a + K_A$, where

$$a(z) = s_A(z) + \sum_{k=1}^{m} a_k(z) v_k^2(z).$$

Example

The Toeplitz operator T_{χ_+} is self-adjoint and $\mathrm{sp}\, T_{\chi_+} = [0, 1]$.
By functional calculus, for each $f \in C([0, 1])$, the operator $f(T_{\chi_+})$ is well defined and belongs to the C^*-algebra generated by T_{χ_+}.

Example

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By functional calculus, for each $f \in C([0,1])$, the operator $f(T_{\chi_+})$ is well defined and belongs to the C^*-algebra generated by T_{χ_+}.

For any $\alpha \in (0,1)$, introduce

$$f_\alpha(x) = x^{2(1-\alpha)} \frac{(1-x)^{2\alpha} - x^{2\alpha}}{(1-x) - x}, \qquad x \in [0,1].$$

Then

$$f_\alpha(T_{\chi_+}) = T_{\chi_{[0,\alpha\pi]}}.$$

Toeplitz sum of products

Example

Let $p(x) = \sum_{k=1}^{n} a_k x^k$ be a polynomial of degree $n \geq 2$. Then the bounded operator $p(T_{\chi_+})$ is not a Toeplitz operator.

Toeplitz sum of products

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Let $p(x) = \sum_{k=1}^{n} a_k x^k$ be a polynomial of degree $n \geq 2$. Then the bounded operator $p(T_{\chi_+})$ is not a Toeplitz operator.

Corollary

Let

$$A = \sum_{i=1}^{p} \prod_{j=1}^{q_i} T_{a_{i,j}} \in \mathcal{T}(PC(\overline{\mathbb{D}}, T)).$$

Then A is a compact perturbation of a Toeplitz operator if and only if A is a compact perturbation an initial generator T_a, for some $a \in PC(\overline{\mathbb{D}}, T)$.

Structure

- Each operator $A \in \mathcal{T}(PC(\overline{\mathbb{D}}, T))$ admits a transparent canonical representation

$$A = T_{s_A} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} + K.$$

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- All initial generators T_a, $a \in PC(\overline{\mathbb{D}}, T)$ are Toeplitz operators.

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- All initial generators T_a, $a \in PC(\overline{\mathbb{D}}, T)$ are Toeplitz operators.
- None of the (non trivial) elements

$$\sum_{i=1}^{p} \prod_{j=1}^{q_i} T_{a_{i,j}},$$

is a compact perturbation of a Toeplitz operator.

Structure

- The uniform closure contains a huge amount of Toeplitz operators, with bounded and even unbounded symbols, which are drastically different from the initial generators.

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- All these Toeplitz operators are uniform limits of sequences of non-Toeplitz operators.
- The uniform closure contains as well many non-Toeplitz operators.

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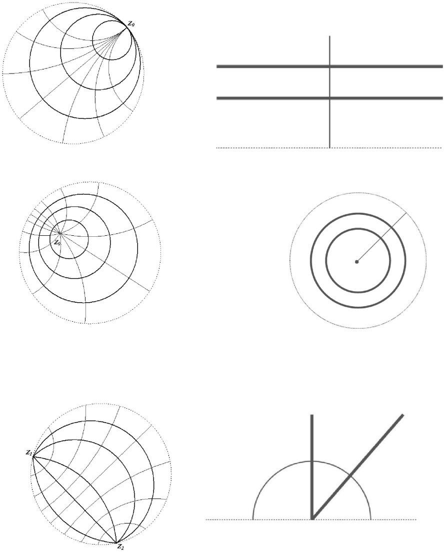
Model cases



Model Maximal Commutative Subgroups

- Elliptic: $\mathbb{T}$, with $z \in \mathbb{D} \longmapsto tz \in \mathbb{D}, \quad t \in \mathbb{T}$,
- Hyperbolic: $\mathbb{R}_+$, with $z \in \Pi \longmapsto rz \in \Pi, \quad r \in \mathbb{R}_+$,
- Parabolic: $\mathbb{R}$, with $z \in \Pi \longmapsto z + h \in \Pi, \quad h \in \mathbb{R}$.

Unit ball

We consider the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$,

$$\mathbb{B}^n = \{z = (z_1, ..., z_n) \in \mathbb{C}^n : \ |z|^2 = |z_1|^2 + ... + |z_n|^2 < 1\}.$$

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For each $\lambda \in (-1, \infty)$, introduce the measure

$$d\mu_\lambda(z) = c_\lambda \, (1 - |z|^2)^\lambda \, dv(z),$$

where $dv(z) = dx_1 dy_1 ... dx_n dy_n$ and

$$c_\lambda = \frac{\Gamma(n + \lambda + 1)}{\pi^n \Gamma(\lambda + 1)}.$$

The (weighted) Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ is the subspace of $L_2(\mathbb{B}^n, d\mu_\lambda)$ consisting of functions analytic in $\mathbb{B}^n$.
The orthogonal Bergman projection has the form

$$(B_{\mathbb{B}^n}\varphi)(z) = \int_{\mathbb{B}^n} \varphi(\zeta) \, \frac{(1 - |\zeta|^2)^\lambda}{(1 - z \cdot \overline{\zeta})^{n+\lambda+1}} \, c_\lambda \, dv(\zeta).$$

Unbounded realizations

The standard unbounded realization of the unit disk $\mathbb{D}$ is the upper half-plane

$$\Pi = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}.$$

The standard unbounded realization of the unit ball $\mathbb{B}^n$ is the Siegel domain in $\mathbb{C}^n$

$$D_n = \{z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \ \operatorname{Im} z_n - |z'|^2 > 0\},$$

where we use the following notation for the points of $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$:

$$z = (z', z_n), \quad \text{where} \ \ z' = (z_1, ..., z_{n-1}) \in \mathbb{C}^{n-1}, \ \ z_n \in \mathbb{C}.$$

Model Maximal Commutative Subgroups

- Quasi-elliptic: $\mathbb{T}^n$, for each $t = (t_1, \dots, t_n) \in \mathbb{T}^n$:
$z = (z_1, \dots, z_n) \in \mathbb{B}^n \mapsto tz = (t_1 z_1, \dots, t_n z_n) \in \mathbb{B}^n$;
- Quasi-hyperbolic: $\mathbb{T}^{n-1} \times \mathbb{R}_+$, for each $(t, r) \in \mathbb{T}^{n-1} \times \mathbb{R}_+$:
$(z', z_n) \in D_n \longmapsto (r^{1/2} t z', r z_n) \in D_n$;
- Quasi-parabolic: $\mathbb{T}^{n-1} \times \mathbb{R}$, for each $(t, h) \in \mathbb{T}^{n-1} \times \mathbb{R}$:
$(z', z_n) \in D_n \longmapsto (tz', z_n + h) \in D_n$;
- Nilpotent: $\mathbb{R}^{n-1} \times \mathbb{R}$, for each $(b, h) \in \mathbb{R}^{n-1} \times \mathbb{R}$:
$(z', z_n) \in D_n \mapsto (z' + b, z_n + h + 2iz' \cdot b + i|b|^2) \in D_n$;
- Quasi-nilpotent: $\mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$, $0 < k < n-1$,
for each $(t, b, h) \in \mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$:
$(z', z'', z_n) \in D_n \longmapsto (tz', z'' + b, z_n + h + 2iz'' \cdot b + i|b|^2) \in D_n$.

Classification Theorem

Theorem

Given any maximal commutative subgroup G of biholomorphisms of the unit ball $\mathbb{B}^n$, denote by $\mathcal{A}_G$ the set of all $L_\infty(\mathbb{B}^n)$-functions which are invariant under the action of G.
Then the C^*-algebra generated by Toeplitz operators with symbols from $\mathcal{A}_G$ is commutative on each weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$, $\lambda \in (-1, \infty)$.

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Then the C^*-algebra generated by Toeplitz operators with symbols from $\mathcal{A}_G$ is commutative on each weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$, $\lambda \in (-1, \infty)$.

The result can be alternatively formulated in terms of the so-called Lagrangian frames, the multidimensional analog of pencils of geodesics and cycles of the unit disk.

It was firmly expected that the situation for the unit ball is pretty much the same as in the case of the unit disk, that is:

The above algebras exhaust all possible algebras of Toeplitz operators on the unit ball which are commutative on each weighted Bergman space.

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The above algebras exhaust all possible algebras of Toeplitz operators on the unit ball which are commutative on each weighted Bergman space.

But:

It turns out that there exist many other Banach algebras generated by Toeplitz operators which are commutative on each weighted Bergman space, non of them is a C^*-algebra, and for $n = 1$ all of them collapse to known commutative C^*-algebras of the unit disk.