INVERSE FUNCTION THEOREM

I use $df_{\mathbf{x}}$ for the linear transformation that is the differential of f at $\mathbf{x}$.

Definition 1. Suppose $S \subseteq \mathbb{R}^n$ is open, $\mathbf{a} \in S$, and $f : S \to \mathbb{R}^n$ is a function. We say f is *locally invertible around* $\mathbf{a}$ if there is an open set $A \subseteq S$ containing $\mathbf{a}$ so that $f(A)$ is open and there is a function $g : f(A) \to A$ so that, for all $x \in A$ and $y \in f(A)$,

$$g(f(x)) = x, \qquad f(g(y)) = y.$$

Clearly, it suffices to have $f(A)$ open and f one-to-one on the open set A. It is important to note how f^{-1} depends on the choice of A. If B another open set and $h : f(B) \to B$ is an inverse for f on B, then on $A \cap B$, h and g agree. So changing the set A may change the domain of f^{-1} but not the value of $f^{-1}(\mathbf{x})$ for any point $\mathbf{x}$.

Definition 2. If $S \subseteq \mathbb{R}^n$ is open, then $g : S \to \mathbb{R}^m$ is *Lipschitz* if there is a constant K so that

$$\|g(\mathbf{w}) - g(\mathbf{y})\| \leq K\|\mathbf{w} - \mathbf{y}\|.$$

We will need the following result:

Proposition 3. *Linear transformations are Lipschitz. That is, for a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$, there is $M > 0$ so that, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,*

$$\|L\mathbf{x} - L\mathbf{y}\| \leq M\|\mathbf{x} - \mathbf{y}\|.$$

We also need the following result:

Proposition 4. *Let $S \subseteq \mathbb{R}^n$ is open. If function $f : S \to \mathbb{R}$ is continuous and $T \subseteq S$ is a closed and bounded set, then f attains its maximum and minimum on T. That is, there is $t_0, t_1 \in T$ so that, for all $t \in T$,*

$$f(t_0) \leq f(t) \leq f(t_1).$$

Note that f does not need to have an inverse function for $f^{-1}(V)$ to make sense.

Theorem 5 (Local Invertibility)**.** *Let $S \subseteq \mathbb{R}^n$ is open, $\mathbf{a} \in S$, and $f : S \to \mathbb{R}^n$ is C^1. If $df_{\mathbf{a}}$ is invertible, then f is locally invertible around $\mathbf{a}$ and f^{-1} is Lipschitz.*

Lemma 6. *With the same hypotheses as the theorem, there are $\epsilon, c > 0$ so that, for all $\mathbf{x}, \mathbf{z} \in B_\epsilon(\mathbf{a})$,*

$$\|f(\mathbf{x}) - f(\mathbf{z})\| \geq c\|\mathbf{x} - \mathbf{z}\|. \tag{7}$$

and, for all $\mathbf{x} \in B_\epsilon(\mathbf{a})$, $df_{\mathbf{x}}$ is invertible.

Proof of Local Invertibility Theorem. Using the lemma, observe that for $\mathbf{x}, \mathbf{z} \in B_\epsilon(\mathbf{a})$ with $\mathbf{x} \neq \mathbf{z}$,

$$\|f(\mathbf{x}) - f(\mathbf{z})\| \geq c\|\mathbf{x} - \mathbf{z}\| > 0$$

and so $f(\mathbf{x}) \neq f(\mathbf{z})$, i.e. f is one-to-one on $B_\epsilon(\mathbf{a})$. Thus, there is a function $f^{-1} : f(B_\epsilon(\mathbf{a})) \to B_\epsilon(\mathbf{a})$. Moreover, for $\mathbf{w}, \mathbf{y} \in f(B_\epsilon(\mathbf{a}))$, there are $\mathbf{x}, \mathbf{z} \in B_\epsilon(\mathbf{a})$ with $\mathbf{w} = f(\mathbf{x})$ and $\mathbf{y} = f(\mathbf{z})$. Using (7),

$$\|\mathbf{w} - \mathbf{y}\| \geq c\|f^{-1}(\mathbf{w}) - f^{-1}(\mathbf{y})\|.$$

This shows f^{-1} is Lipschitz (with constant $1/c$) and so is continuous.

To see that $f(B_\epsilon(\mathbf{a}))$ is open, fix $\mathbf{v}$ in this set. There is $\mathbf{x} \in B_\epsilon(\mathbf{a})$ with $f(\mathbf{x}) = \mathbf{v}$. Choose $s > 0$ so that $\overline{B_s(\mathbf{x})}$ is contained in $B_\epsilon(\mathbf{a})$. Then $S = \{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| = s\}$, the boundary of $\overline{B_s(\mathbf{x})}$, is a closed and bounded set. Since f is continuous, the image, $f(S)$, is also closed and bounded. By the proposition, there is $\mathbf{y}_0 \in S$ so that the function $\mathbf{z} \mapsto \|f(\mathbf{z}) - \mathbf{v}\|$ attains its minimum. That is, for all $\mathbf{y} \in S$,

$$\|f(\mathbf{y}) - \mathbf{v}\| \geq \|f(\mathbf{y}_0) - \mathbf{v}\|.$$

As f is one-to-one, $\mathbf{v}$ is not in $f(S)$; so $d = \|f(\mathbf{y}_0) - \mathbf{v}\| > 0$.

We shall show that $B_{d/2}(\mathbf{v})$ is contained in $f(B_\epsilon(\mathbf{a}))$. Let $\mathbf{u} \in B_{d/2}(\mathbf{v})$ and define a function on $\overline{B_s(\mathbf{x})}$ by

$$g(\mathbf{y}) = \|f(\mathbf{y}) - \mathbf{u}\|^2 = (f(\mathbf{y}) - \mathbf{u}) \cdot (f(\mathbf{y}) - \mathbf{u}).$$

Observe that g is C^1 because f is and by previous work

$$Dg_{\mathbf{y}}(\mathbf{h}) = 2\big(Df_{\mathbf{y}})(\mathbf{h})\big) \cdot \big(f(\mathbf{y}) - \mathbf{u})\big).$$

Since $\overline{B_s(\mathbf{x})}$ is a closed and bounded set and g is continuous, the proposition guarantees that g attains its minimum value. Observe that at every point of S,

$$g(\mathbf{y}) = \|f(\mathbf{y}) - \mathbf{u}\|^2 \geq (\|f(\mathbf{y}) - \mathbf{v}\| - \|\mathbf{v} - \mathbf{u}\|)^2 \geq \left(d - \frac{d}{2}\right)^2 = \frac{d^2}{4},$$

while

$$g(\mathbf{x}) = \|\mathbf{v} - \mathbf{u}\|^2 < \frac{d^2}{4}.$$

Hence the minimum of g occurs at some interior point $\mathbf{y}_0$. So by previous work, $Dg(\mathbf{y}_0) = 0$. But $df(\mathbf{y}_0)$ is invertible by the lemma, so $f(\mathbf{y}_0) - \mathbf{u} = 0$; that is, $f(\mathbf{y}_0) = \mathbf{u}$. Therefore $f(B_\epsilon(\mathbf{a}))$ is open.

Proof of Lemma. Let $T = (df_{\mathbf{a}})^{-1}$. By the proposition above, there is $M > 0$ so that

$$\|T\mathbf{u} - T\mathbf{v}\| \leq M\|\mathbf{u} - \mathbf{v}\|.$$

Letting $\mathbf{u} = T^{-1}(\mathbf{x} - \mathbf{a})$ and $\mathbf{v} = T^{-1}(\mathbf{y} - \mathbf{a})$, (so $\mathbf{u} = df_{\mathbf{a}}(\mathbf{x} - \mathbf{a})$), we have

$$\|df_{\mathbf{a}}(\mathbf{x} - \mathbf{a}) - df_{\mathbf{a}}(\mathbf{y} - \mathbf{a})\| \leq \frac{1}{M}\|\mathbf{x} - \mathbf{y}\|.$$

Define $E : S \to \mathbb{R}^n$ by $E(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{a}) - df_{\mathbf{a}}(\mathbf{x} - \mathbf{a})$. Since f is C^1 and linear transformations are infinitely differentiable, E is C^1. Notice that

$$dE_{\mathbf{a}}(\mathbf{h}) = df_{\mathbf{a}}(\mathbf{h}) - df_{\mathbf{a}}(\mathbf{h}) = 0.$$

In particular, if $E = (E_1, \ldots, E_n)$, then by the continuity of $d(E_i)_{\mathbf{a}}$ there is some $\epsilon > 0$ so that

$$\|d(E_i)_{\mathbf{z}}\| \leq \frac{1}{2M\sqrt{n}},$$

for $i = 1, \ldots, n$ and all $\mathbf{z} \in B_\epsilon(\mathbf{a})$.

Suppose that $\mathbf{x}, \mathbf{z} \in B_\epsilon(\mathbf{a})$. Then, for each i, by Taylor's Theorem with linear remainder term, there is $\mathbf{c}_i \in L[\mathbf{x}, \mathbf{z}] \subset B_\epsilon(\mathbf{a})$ so that

$$|E_i(\mathbf{x}) - E_i(\mathbf{z})| = |d(E_i)_{\mathbf{c}_i}(\mathbf{x} - \mathbf{z})| \leq \frac{1}{2M\sqrt{n}}\|\mathbf{x} - \mathbf{z}\|.$$

and so

$$\begin{aligned}
\|E(\mathbf{x}) - E(\mathbf{z})\|^2 &= \sum_{i=1}^{n} |E_i(\mathbf{x}) - E_i(\mathbf{z})|^2 \\
&\le \sum_{i=1}^{n} \left(\frac{1}{2M\sqrt{n}}\right)^2 \|\mathbf{x} - \mathbf{z}\|^2 \\
&= \left(\frac{1}{2M}\right)^2 \|\mathbf{x} - \mathbf{z}\|^2.
\end{aligned}$$

Thus, $\|E(\mathbf{x}) - E(\mathbf{z})\| \le \|\mathbf{x} - \mathbf{z}\|/(2M)$.

As $f(\mathbf{x}) - f(\mathbf{z}) = E(\mathbf{x}) - E(\mathbf{z}) - (df_{\mathbf{a}}(\mathbf{x} - \mathbf{a}) - df_{\mathbf{a}}(\mathbf{z} - \mathbf{a}))$,

$$\begin{aligned}
\|f(\mathbf{x}) - f(\mathbf{z})\| &\ge \|df_{\mathbf{a}}(\mathbf{x} - \mathbf{a}) - df_{\mathbf{a}}(\mathbf{z} - \mathbf{a})\| - \|E(\mathbf{x}) - E(\mathbf{z})\| \\
&\ge \frac{1}{M}\|\mathbf{x} - \mathbf{z}\| - \frac{1}{2M}\|\mathbf{x} - \mathbf{z}\| = \frac{1}{2M}\|\mathbf{x} - \mathbf{z}\|.
\end{aligned}$$

The proves (7) with $c = 1/(2M)$.

Finally, to see that $df_{\mathbf{x}}$ in invertible for each $\mathbf{x} \in B_\epsilon(\mathbf{a})$, observe that

$$dE_{\mathbf{x}}(\mathbf{z} - \mathbf{x}) = df_{\mathbf{x}}(\mathbf{z} - \mathbf{x}) - df_{\mathbf{a}}(\mathbf{z} - \mathbf{x}).$$

If there was $\mathbf{z}$ so that $df_{\mathbf{x}}(\mathbf{z} - \mathbf{x}) = 0$, then $dE_{\mathbf{x}}(\mathbf{z} - \mathbf{x}) = -df_{\mathbf{a}}(\mathbf{z} - \mathbf{x})$. On the other hand, we have that

$$\|df_{\mathbf{a}}(\mathbf{z} - \mathbf{x})\| \ge \frac{1}{M}\|\mathbf{z} - \mathbf{x}\|, \qquad \|dE_{\mathbf{x}}(\mathbf{z} - \mathbf{x})\| \le \frac{1}{2M}\|\mathbf{z} - \mathbf{x}\|.$$

This contradiction shows that $df_{\mathbf{x}}$ must be invertible.

Recall that we proved that a function g is differentiable at $\mathbf{c}$ if and only if there is a linear transformation L and a function ϵ so that $\lim_{\mathbf{x} \to \mathbf{c}} \epsilon(\mathbf{x}) = 0$ and

$$g(\mathbf{x}) = g(\mathbf{c}) + L(\mathbf{x} - \mathbf{c}) + \epsilon(\mathbf{x})\|\mathbf{x} - \mathbf{c}\|.$$

In this case, L is $dg_{\mathbf{c}}$.

Theorem 8 (Inverse Function Theorem)**.** *Let $S \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in S$, and $f : S \to \mathbb{R}^n$ is C^1. If $df_{\mathbf{a}}$ is invertible, then f^{-1} is differentiable at $\mathbf{b} = f(\mathbf{a})$ and*

$$d(f^{-1})_{\mathbf{b}} = \left(df_{f^{-1}(\mathbf{b})}\right)^{-1}.$$

Proof. Since f is differentiable at $\mathbf{a}$, there is a function $\epsilon : S \to \mathbb{R}^n$ with $\lim_{\mathbf{x} \to \mathbf{a}} \epsilon(\mathbf{x}) = 0$ and

$$f(\mathbf{x}) = f(\mathbf{a}) + df_{\mathbf{a}}(\mathbf{x} - \mathbf{a}) + \epsilon(\mathbf{x})\|\mathbf{x} - \mathbf{a}\|.$$

Since f is locally invertible around $\mathbf{a}$, there is some open set A containing $\mathbf{a}$ on which f is one-to-one and f^{-1} is Lipschitz on the open set $f(A)$.

For $\mathbf{x} \in A$, there is $\mathbf{y} \in f(A)$ with $\mathbf{x} = f^{-1}(\mathbf{y})$. Using this and $\mathbf{a} = f^{-1}(\mathbf{b})$, we have

$$f(f^{-1}(\mathbf{y})) = f(f^{-1}(\mathbf{b})) + df_{\mathbf{a}}(f^{-1}(\mathbf{y}) - f^{-1}(\mathbf{b})) + \epsilon(f^{-1}(\mathbf{y}))\|f^{-1}(\mathbf{y}) - f^{-1}(\mathbf{b})\|.$$

Using the inverse function identities and moving $\mathbf{b}$ over, we have

$$\mathbf{y} - \mathbf{b} = df_{\mathbf{a}}(f^{-1}(\mathbf{y}) - f^{-1}(\mathbf{b})) + \epsilon(f^{-1}(\mathbf{y}))\|f^{-1}(\mathbf{y}) - f^{-1}(\mathbf{b})\|.$$

Applying $(df_{\mathbf{a}})^{-1}$ to this equation and using the linearity of $(df_{\mathbf{a}})^{-1}$, we have

$$(df_{\mathbf{a}})^{-1}(\mathbf{y}-\mathbf{b}) = f^{-1}(\mathbf{y}) - f^{-1}(\mathbf{b}) + (df_{\mathbf{a}})^{-1}(\epsilon(f^{-1}(\mathbf{y})))\|f^{-1}(\mathbf{y}) - f^{-1}(\mathbf{b})\|.$$

Then we can rearrange the previous equation to obtain

$$f^{-1}(\mathbf{y}) = f^{-1}(\mathbf{b}) + (df_{\mathbf{a}})^{-1}(\mathbf{y}-\mathbf{b}) + \eta(\mathbf{y})\|\mathbf{y}-\mathbf{b}\|.$$

if we define a new function η on $f(A)$ by letting $\eta(\mathbf{b}) = 0$ and otherwise

$$\eta(\mathbf{y}) = \frac{-(df_{\mathbf{a}})^{-1}(\epsilon(f^{-1}(\mathbf{y})))\|f^{-1}(\mathbf{y}) - f^{-1}(\mathbf{b})\|}{\|\mathbf{y}-\mathbf{b}\|}.$$

To show that f^{-1} is differentiable at $\mathbf{b}$ and $d(f^{-1})_{\mathbf{b}}$ is $(df_{\mathbf{a}})^{-1}$, it suffices to show that

$$\lim_{\mathbf{y}\to\mathbf{b}} \eta(\mathbf{y}) = 0.$$

As f^{-1} is Lipschitz, there is a constant $K > 0$ so that

$$\frac{\|f^{-1}(\mathbf{y}) - f^{-1}(\mathbf{b})\|}{\|\mathbf{y}-\mathbf{b}\|} \le K$$

for all $\mathbf{y} \in f(A)$. So it suffices to prove that

$$\lim_{\mathbf{y}\to\mathbf{b}} -(df_{\mathbf{a}})^{-1}(\epsilon(f^{-1}(\mathbf{y}))) = 0.$$

Now, as $\mathbf{y} \to \mathbf{b}$, $f^{-1}(\mathbf{y}) \to f^{-1}(\mathbf{b}) = \mathbf{a}$. By our choice of the function ϵ, as $f^{-1}(\mathbf{y}) \to \mathbf{a}$, $\epsilon(f^{-1}(\mathbf{y})) \to 0$. Since the linear transformation $(df_{\mathbf{a}})^{-1}$ is continuous, we have the claimed limit.

This concludes the proof.

Corollary 9. *f^{-1} is C^1 on its domain.*

This is very rough. Notice first that since f^{-1} is uniquely defined on its domain, call it A, f is locally invertible at each point of A. By the lemma, we may assume $df_{\mathbf{a}}$ is invertible for each $\mathbf{a} \in A$. By the inverse function theorem, we have that $d(f^{-1})_{\mathbf{b}} = (df_{f^{-1}(\mathbf{b})})^{-1}$ for each $\mathbf{b} \in f(A)$.

To see that this function is continuous, observe first that f^{-1} is continuous; second, that the map $\mathbf{x} \mapsto df_{\mathbf{x}}$ is continuous; and third, that matrix inversion is continuous. As a composition of three continuous operations, $d(f^{-1})_{\mathbf{b}}$ is a continuous function of $\mathbf{b}$.