Math 107, Sect. 250 **Answers to Quiz # 5** **22 Feb '07**

. 5 1. Does the sequence $\{a_n\}_{n=1}^{\infty}$ converge or diverge? If it converges, find to what the sequence converges.

$$a_n = \frac{n + \sin(n)}{2n - 1}$$

Solution. We compute the following limit:

$$\lim_{n\to+\infty} a_n \quad = \quad \lim_{n\to+\infty} \frac{n+\sin(n)}{2n-1} \; .$$

Since $-1 \leq \sin(n) \leq 1$, we have

$$\frac{n-1}{2n-1} \leq \frac{n+\sin(n)}{2n-1} \leq \frac{n+1}{2n-1} \; .$$

Note that

$$\lim_{n\to+\infty} \frac{n-1}{2n-1} = \lim_{n\to+\infty} \frac{1-1/n}{2-1/n} = \frac{1}{2} \quad \text{and} \quad \lim_{n\to+\infty} \frac{n+1}{2n-1} = \lim_{n\to+\infty} \frac{1+1/n}{2-1/n} = \frac{1}{2} \; .$$

By the Squeeze Theorem, we have:

$$\lim_{n\to+\infty} a_n = \frac{1}{2} \; .$$

Hence the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 1/2.

. 5 2. Does the following series converge or diverge? If it converges, find its sum.

$$\sum_{n=1}^{\infty} (\arctan(n) - \arctan(n+1))$$

Solution. This is a telescoping series. The k-th partial sum s_k is given by:

$$\begin{aligned}
s_k \quad &= \quad \sum_{n=1}^{k} (\arctan(n) - \arctan(n+1)) = \\
&= \quad (\arctan(1) - \arctan(2)) + (\arctan(2) - \arctan(3)) + \cdots \\
&\quad\;\; + (\arctan(k-1) - \arctan(k)) + (\arctan(k) - \arctan(k+1)) = \\
&= \quad \arctan(1) - \arctan(k+1) \; .
\end{aligned}$$

Thus

$$\lim_{k\to+\infty} s_k = \lim_{k\to+\infty} (\arctan(1) - \arctan(k+1)) = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4} \; .$$

Hence the series converges and its sum is $-\pi/4$:

$$\sum_{n=1}^{\infty} (\arctan(n) - \arctan(n+1)) = -\frac{\pi}{4} \; .$$