Math 106 **Answers to Test #1** **11 Feb 08**

. 12 1. A projectile is launched vertically. Its height above the ground is given by $y = 192t - 16t^2$, where y is the height in feet and t is the time since the launch, in seconds.

(a) What is $\dfrac{dy}{dt}$ when $t = 5$? Include units.

(b) How long is the projectile aloft?

(c) What is its maximum height?

Solution. For part (a), we have

$$\frac{dy}{dt} = 192 - 32t$$

and so

$$\left.\frac{dy}{dt}\right|_{t=5} = 192 - 32 \cdot 5 = 32 \text{ ft/sec.}$$

For part (b), the time it is aloft is from the launch, $t = 0$, until hits the ground again, i.e., when $y = 0$. So we solve $y = 0$, to get $192t - 16t^2 = 0$ or $t(192 - 16t) = 0$ so $t = 0$ (which we already knew about) and $196 - 16t = 0$ or $t = 192/16 = 12$ seconds.

For part (c), the maximum height occurs when the derivative is zero, so first we solve $\dfrac{dy}{dt} = 0$, that is, $192 - 32t = 0$ or $t = 192/32 = 6$ seconds. Thus, the maximum height is

$$y = 192 \cdot 6 - 16 \cdot 6^2 = 576 \text{ feet.}$$

. 8 2. Find the derivative of $f(x) = \dfrac{x^2 + 2e^x}{e^x - x^3}$. Do not simplify your answer.

Solution. Using the quotient rule, we have

$$f'(x) = \frac{(e^x - x^3)(2x + 2e^x) - (x^2 + 2e^x)(e^x - 3x^2)}{e^x - x^3)^2}$$

. 16 3. For the graph of $y = f(x)$ given below, answer the following questions. (Show all relevant work, of course.)

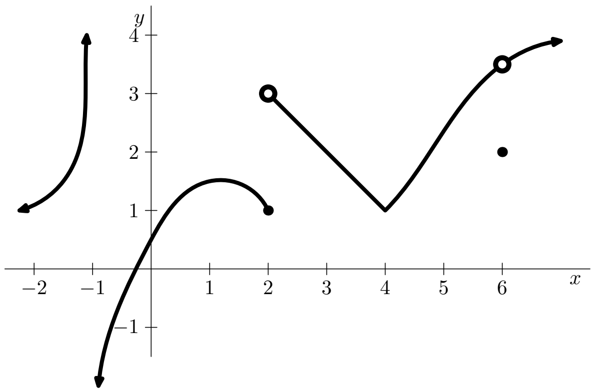
(a) What is $f'(3)$?

(b) What are $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$?

(c) What is the average rate of change of $f(x)$ from $x = 0$ to $x = 2$?

(d) List all x-values where $f(x)$ is not continuous.

(e) List all x-values where $f(x)$ is not differentiable.



Solution. We have $f'(3) = -1$, $\lim_{x\to 2^-} f(x) = 1$, $\lim_{x\to 2^+} f(x) = 3$.
For (c), the average rate of change is

$$\frac{f(2) - f(0)}{2 - 0} = \frac{1 - 1/2}{2} = \frac{1}{4}.$$

For (d), the discontinuities are at $x = -1$, $x = 2$ and $x = 6$.
For (e), f is not differentiable at $x = -1$, $x = 2$, $x = 4$, and $x = 6$.

. 13 4. Suppose that u and v are functions of x that are differentiable at $x = 1$ and that

$$u(1) = -1, \quad u'(1) = 4, \quad v(1) = 3, \quad v'(1) = 2.$$

Find the values of the following derivatives at $x = 1$:

$$\text{(a) } \frac{d}{dx}(uv), \qquad \text{(b) } \frac{d}{dx}\left(\frac{u}{v}\right).$$

Solution. For (a), we use the product rule, to get

$$\left.\frac{d}{dx}(uv)\right|_{x=1} = u(1)v'(1) + u'(1)v(1) = (-1)\cdot 2 + 4 \cdot 3 = -2 + 12 = 10.$$

and for (b), we use the quotient rule to obtain

$$\left.\frac{d}{dx}\left(\frac{u}{v}\right)\right|_{x=0} = \frac{v(0)u'(0) - u(0)v'(0)}{v(0)^2} = \frac{3\cdot 4 - (-1)\cdot 2}{3^2} = \frac{12 + 2}{9} = \frac{14}{9}.$$

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5. Find the horizontal and vertical asymptotes of $y = \dfrac{x^2 - 25}{x^2 + 2x - 15}$ and sketch its graph, including intercepts. (Be sure to justify your asymptotes).

Solution. To find the horizontal asymptotes, we compute

$$\lim_{x\to\infty} \frac{x^2 - 25}{x^2 + 2x - 15} = \lim_{x\to\infty} \frac{1 - 25/x^2}{1 - 2/x - 15/x^2} = 1.$$

and the calculation for $\lim\limits_{x\to-\infty} \dfrac{x^2 + 1}{2x^2 - x - 6} = 1$ is exactly the same. So the line $y = 1$ is a horizontal asymptote.

To find the vertical asymptotes, we first factor the denominator, giving $x^2 + 2x - 15 = (x + 5)(x - 3)$. So $x = -5$ and $x = 3$ are *possible* vertical asymptotes. However, the numerator is zero at $x = -5$, so this point may not be an asymptote. In fact,

$$\frac{x^2 - 25}{x^2 + 2x - 15} = \frac{(x - 5)(x + 5)}{(x + 5)(x - 3)} = \frac{x - 5}{x - 3}.$$

Thus, there is a removable discontinuity at $x = -5$ and a vertical asymptote at $x = 3$.

To sketch the graph we need to know the signs of the one-sided limits, or the signs of the function. We give the answer using limits:

For $x \to 3^+$, we have $x - 3$ is positive and $x - 5$ is negative, so $\lim\limits_{x\to3^+} f(x) = \lim\limits_{x\to3^+} \dfrac{x - 5}{x - 3} = -\infty$. For $x \to 3^-$, we have $x - 3$ is negative and $x - 5$ is (still) negative so $\lim\limits_{x\to3^+} f(x) = \lim\limits_{x\to3^+} \dfrac{x - 5}{x - 3} = +\infty$.

Finally, we compute the intercepts. For the y-axis intercept, let $x = 0$, giving $y = (0-5)/(0-3) = 5/3$. For the x-axis intercept, solve $(x-5)/(x-3) = 0$, that is, $x - 5 = 0$ or $x = 5$.

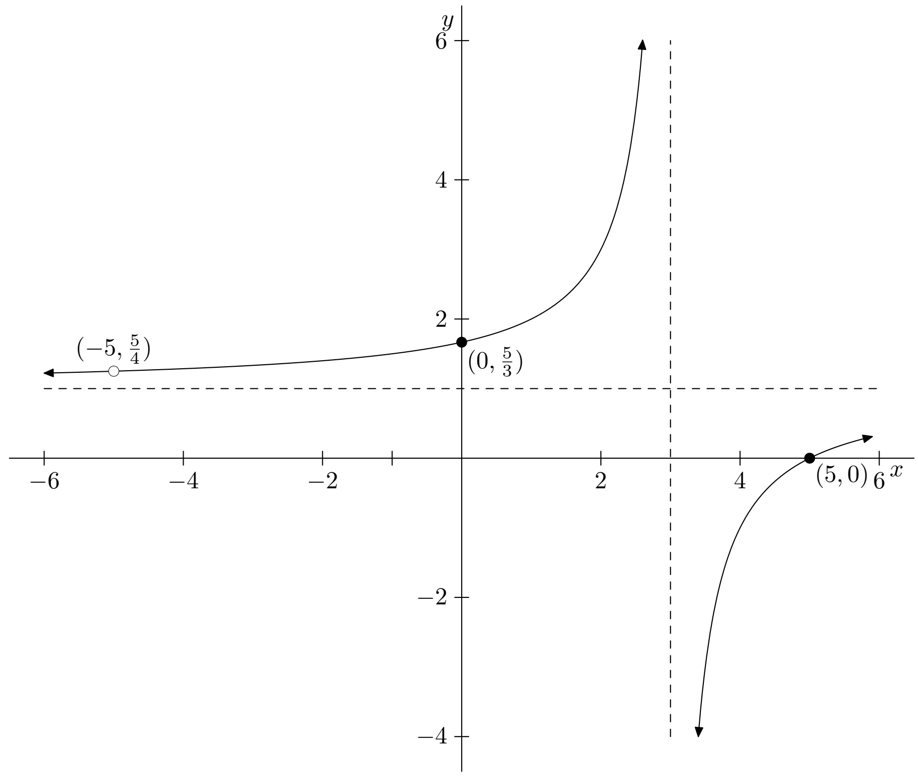

Figure 1: Graph of $f(x) = \dfrac{x^2 - 25}{x^2 + 2x - 15}$.

. 18 6. Evaluate the following limits. (You can use the simplest limit laws, like those for sums and products, without comment but you should clearly indicate when you are using more complicated properties.)

$$\text{(a)} \lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16}, \qquad \text{(b)} \lim_{y \to 0} \frac{\sin y}{\sin 3y}, \qquad \text{(c)} \lim_{z \to -2^+} \frac{x^2 + 1}{x^2 - 4}.$$

Solution. For (a),

$$\lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16} = \lim_{x \to 16} \frac{\sqrt{x} - 4}{(\sqrt{x} - 4)(\sqrt{x} + 4)} = \lim_{x \to 16} \frac{1}{\sqrt{x} + 4} = \frac{1}{8}.$$

where the evaluation of the limit is possible by the quotient limit law, since the denominator is not zero.

For (b), we use the limit $\lim_{y \to 0} \frac{\sin y}{y} = 1$ and the same limit with y replaced by $3y$. Thus,

$$\frac{\sin y}{\sin 3y} = \frac{\sin y}{y} \frac{3y}{\sin 3y} \frac{1}{3}.$$

Taking limits, we have

$$\lim_{y \to 0} \frac{\sin y}{\sin 3y} = \lim_{y \to 0} \frac{\sin y}{y} \frac{3y}{\sin 3y} \frac{1}{3} = \left(\lim_{y \to 0} \frac{\sin y}{y} \right) \left(\lim_{y \to 0} \frac{3y}{\sin 3y} \right) \frac{1}{3} = 1 \cdot 1 \cdot \frac{1}{3} = \frac{1}{3}.$$

For (c), first observe that, when $x = -2$, $x^2 + 1 = 5$ and $x^2 - 4 = 0$, so the value of the limit is either ∞ or $-\infty$. To figure out which, we compute

$$\lim_{z \to -2^+} \frac{x^2 + 1}{x^2 - 4} = \lim_{z \to -2^+} \frac{x^2 + 1}{(x - 2)(x + 2)}.$$

Now if x is a bit more than -2, say -1.99, then $x^2 + 1$ is about $5 > 0$, while $x - 2$ is about -4 and $x + 2$ is a bit more than zero. Since we have one negative and two postive quantities, the overall expression is negative and so

$$\lim_{z \to -2^+} \frac{x^2 + 1}{x^2 - 4} = -\infty.$$

. 15 7. Using the definition, find the derivative of $f(x) = x^2 + 1$. Find the equation of the tangent line at $x = 3$.

Check your derivative using the "rules".

Solution. By definition

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{((x + h)^2 + 1) - (x^2 + 1)}{h} \\
&= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 + 1) - x^2 - 1}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} \frac{h(2x + h)}{h} \\
&= \lim_{h \to 0} 2x + h = 2x.
\end{aligned}$$

To check this answer using the power rule and the sum rule, compute $f'(x) = 2x + 0 = 2x$. so our answer is correct.

At $x = 3$, $f(3) = 3^2 + 1 = 10$ and $f'(3) = 2 \cdot 3 = 6$. so this is the line through $(3, 10)$ with slope 6. Using the point slope form, it is $y - 10 = 6(x - 3)$ which simplifies to $y = 6x - 8$.