RIEMANN SUM EXAMPLE

We find and simplify the Riemann Sum formula for $f(x) = 3 + 2x - x^2$ on $[0, 3]$ using n equal subintervals and the lefthand rule.

$$\begin{aligned}
\text{Sum} &= f(0)\frac{3}{n} + f\left(\frac{3}{n}\right)\frac{3}{n} + f\left(\frac{6}{n}\right)\frac{3}{n} + f\left(\frac{9}{n}\right)\frac{3}{n} + \cdots + f\left(\frac{3n-3}{n}\right)\frac{3}{n} \\
&= \sum_{i=1}^{n} f\left(\frac{3(i-1)}{n}\right)\frac{3}{n} \\
&= \sum_{i=1}^{n} \left(3 + \frac{6(i-1)}{n} - \frac{9(i-1)^2}{n^2}\right)\frac{3}{n} \\
&= \sum_{i=1}^{n} \left(\frac{9}{n} + \frac{18(i-1)}{n^2} - \frac{27(i-1)^2}{n^3}\right) \\
&= \sum_{i=1}^{n} \frac{9}{n} + \sum_{i=1}^{n} \frac{18(i-1)}{n^2} - \sum_{i=1}^{n} \frac{27(i-1)^2}{n^3} \\
&= n\frac{9}{n} + \frac{18}{n^2}\sum_{i=1}^{n}(i-1) - \frac{27}{n^3}\sum_{i=1}^{n}(i-1)^2
\end{aligned}$$

and, letting $j = i - 1$, we get

$$= 9 + \frac{18}{n^2}\sum_{j=0}^{n-1} j - \frac{27}{n^3}\sum_{j=0}^{n-1} j^2$$

and we can leave out the $j = 0$ terms, since they add nothing, to get

$$= 9 + \frac{18}{n^2}\sum_{j=1}^{n-1} j - \frac{27}{n^3}\sum_{j=1}^{n-1} j^2$$

and now we use the formulas with $n - 1$ to get

$$\begin{aligned}
&= 9 + \frac{18}{n^2}\left(\frac{(n-1)n}{2}\right) - \frac{27}{n^3}\left(\frac{(n-1)n(2n-1)}{6}\right) \\
&= 9 + \frac{9(n-1)}{n} - \frac{9(n-1)(2n-1)}{2n^2} \\
&= 9 + \frac{9(n-1)}{n} - \frac{9(2n^2-3n+1)}{2n^2}.
\end{aligned}$$

For $n = 10$, this sum is equal to 9.405, for $n = 100$, it is 9.045 and For $n = 1000$, it is 9.004. Finally, we can take the limit as $n \to +\infty$, to get

$$\lim_{n\to\infty} \text{Sum} = \lim_{n\to\infty} 9 + \frac{9(n-1)}{n} - \frac{9(2n^2-3n+1)}{2n^2} = 9 - 1 + 1 = 9.$$