Math 817–818 Qualifying Exam

June 2011

Rules of the game:

(a) Solve *two* problems from each of the three sections, for a total of *six*. If you work on more than two problems from a given section, indicate *clearly* which problems you want counted.

For problems with multiple parts you may assume the results of earlier parts, even if you have not solved them.

If you have doubts about the wording of a problem, please ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

(b) **Justify all of your answers**.

(c) Each problem is worth 20 points. For problems with multiple parts, bold numbers in [**brackets**] indicate the number of points assigned for that part.

Section I: Groups

(1) Prove that no group of order 150 is simple.

(2) Let G be a finite group.

(a) [**5**] If N is a normal subgroup of G and $|N| = 2$, prove that N is contained in the center $\mathrm{Z}(G)$ of G.

(b) [**15**] Suppose that $|\,\mathrm{Z}(G)|$ is odd and that G contains a non-trivial simple subgroup H with $[G : H] = 2$. Prove that H is the only non-trivial proper normal subgroup of G.

(3) Let H be a normal subgroup of a finite group G, p a prime dividing the order of H, and P a Sylow p-subgroup of H. Prove that $G = H\,\mathrm{N}_G(P)$. [Hint: For $g \in G$, consider the subgroup gPg^{-1}.]

(4) Fix a prime number p, and let A denote the abelian group of all complex roots of unity whose orders are powers of p; that is,

$$A = \{z \in \mathbb{C} \mid z^{p^n} = 1 \text{ for some integer } n \geq 1\}\,.$$

Prove the following statements.

(a) [**5**] Every non-trivial subgroup of A contains the group of p^{th} roots of unity.

(b) [**5**] Every proper subgroup of A is cyclic.

(c) [**5**] If B and C are subgroups of A, then either $B \subseteq C$ or $C \subseteq B$.

(d) [**5**] For each $n \geq 0$ there exists a unique subgroup of A with p^n elements.

Section II: Rings and Fields

(5) Let F be a field, and let $f(x) \in F[x]$. Recall that $f(x)$ is *separable* provided, for every extension field K/F, $f(x)$ has no multiple roots in K. (A *multiple root* is an element $\alpha \in K$ such that $(x-\alpha)^2 \mid f(x)$ in $K[x]$.)

(a) **[13]** Prove that $f(x) \in F[x]$ is separable if and only if $f(x)$ and its derivative $f'(x)$ are relatively prime in $F[x]$.

(b) **[7]** Suppose that $f(x)$ is irreducible and that the degree of $f(x)$ is not a multiple of the characteristic of F. Prove that $f(x)$ is separable.

(6) Let E be the splitting field of the polynomial $x^7 - 12$ over $\mathbb{Q}$. Find $[E : \mathbb{Q}]$, and describe the elements of $\mathrm{Gal}(E/\mathbb{Q})$ explicitly.

(7) Let p be a prime number, let $\mathbb{F}_p$ be the field with p elements, and let n be a positive integer. Prove that $x^{p^n} - x = \prod_{d|n} g_d(x)$, where, for a positive divisor d of n, $g_d(x)$ denotes the product of all monic irreducible polynomials of degree d in $\mathbb{F}_p[x]$. [You may assume basic results on the structure of finite fields and their subfields.]

(8) Let R be a commutative ring with 1. Recall that $r \in R$ is called *nilpotent* if $r^n = 0$ for some integer $n > 0$. Let N be the set of nilpotent elements of R.

(a) **[7]** Show that N is an ideal of R.

(b) **[6]** Show that the ring R/N has no nonzero nilpotent elements.

(c) **[7]** For a polynomial $f(x) \in R[x]$, prove that $f(x)$ is a nilpotent element of $R[x]$ if and only if every coefficient of $f(x)$ is nilpotent.

Section III: Linear Algebra and Modules

Note: This section has just three questions.

(9) Consider the following matrix over $\mathbb{Q}$:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{bmatrix}$$

(a) **[7]** Show that the characteristic and minimal polynomials of A are, respectively, $(x+1)^4$ and $(x+1)^3$.

(b) **[6]** Find the rational canonical form R of A and the Jordan canonical form J of A.

(c) **[7]** Find an invertible matrix P such that $P^{-1}AP = J$.

(10) Let A and B be $n \times n$ matrices with entries in F. Recall that A and B are said to be *similar over* F if there exists an invertible $n \times n$ matrix, with entries in F, such that $B = P^{-1}AP$.

Prove the following statements about matrices A and B with entries in F:

(a) **[10]** If $F \subseteq K$ is a field extension, and A and B are similar over K, then they are similar over F.

(b) **[10]** A is similar over F to its transpose A^{T}.

(11) Let R be a commutative ring with $1 \neq 0$, and let $f\colon R^m \twoheadrightarrow R^n$ be a surjective homomorphism of free R-modules. Prove that $m \geq n$.