Math 932-933, Comprehensive Exam, January, 2002

1. Use the variation of constants formula to solve the IVP
$$x' = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ e^{2t} \\ e^{3t} \end{bmatrix}, \quad x(0) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

2. Let $x(t; a, b)$ denote the solution of the IVP
$$x' = x - x^2, \quad x(a) = b.$$
Without solving the above differential equation, find
$$\frac{\partial x}{\partial b}(t; 0, 1) \quad \text{and} \quad \frac{\partial x}{\partial a}(t; 0, 1).$$
Use one of your answers to approximate $x(t; 0, 1+h)$ for h close to zero. Check your answers for the above partial derivatives by actually solving the given differential equation for $x(t; a, b)$ and then calculating these partial derivatives.

3. Show that if $\Phi(t)$ is a fundamental matrix for the n dimensional vector differential equation $x' = A(t)x$, then $\Psi(t) := \Phi(t)C$, where C is a nonsingular $n \times n$-matrix is also a fundamental matrix. Then prove that all fundamental matrices are of this form.

4. (a) State the Picard-Lindelof Theorem. Maximize the α in the Picard-Lindelof Theorem by choosing the appropriate rectangle Q concerning the IVP $x' = t + x^2, \quad x(0) = 1$.
   (b) State a form of the Ascoli-Arzela Theorem. What happens if you try to apply this theorem to the sequence $\{x_n(t) = \sin(nt)\}_{n=1}^{\infty}$, $t \in [0, \pi]$.

5. Show that the matrix norm induced by the maximum vector norm on $\mathbb{R}^n$ is given by
$$\|A\| = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

6. Apply the LaSalle Invariance Theorem to the system
$$\begin{aligned} x' &= 4y^3 - xy^2 \\ y' &= -3x + x^2 y. \end{aligned}$$
Give a very accurate description of the domain of attraction to the origin. (Hint: Take $V(x, y) = \alpha x^2 + \beta y^4$.)

7. Assume μ_0 is a Floquet multiplier for the Floquet system $x' = A(t)x$, where ω is the smallest positive period of the matrix function $A(t)$.

   (a) Show that there is a nontrivial solution $x_0(t)$ satisfying
$$x_0(t + \omega) = \mu_0 x_0(t), \quad t \in \mathbb{R}.$$
   (b) Show that there is a nontrivial solution of the form
$$x(t) = p(t)e^{\alpha t},$$
where p is a continuously differentiable function on $\mathbb{R}$ which is periodic with period ω and α is a constant.

   (c) Show that if $\mu_0 = -1$ there is a nontrivial solution of the Floquet system which is periodic with period 2ω.