Your Exam ID Code: _____

Real Analysis Comprehensive Examination

Thursday, January 19, 2006, 2:00–6:00p.m., Oldfather 309

• Work 6 of the 8 problems below. • Each problem is worth 20 points. • Write on one side of the paper only.
• Throughout the exam, Lebesgue measure is denoted by m.

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1) Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $f \in L^1(X, \mu)$. Define a function ν on $\mathcal{S}$ by $\nu(A) := \int_A f\, d\mu$. Prove that ν is a signed measure on X. Give formulas for $\nu^+(A), \nu^-(A)$, and $|\nu|(A)$ (in terms of f), being sure to completely justify your formulas.

2) Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose $(Y, \mathcal{B})$ is a measurable space. Let $\varphi : X \to Y$ be a function such that $\varphi^{-1}(E) \in \mathcal{A}$ whenever $E \in \mathcal{B}$.

a) (5 points) For each $E \in \mathcal{B}$, let $\nu(E) = \mu(\varphi^{-1}(E))$. Prove that ν is a measure on $(Y, \mathcal{B})$.

b) (5 points) For each measurable function $f : Y \to \mathbf{R}$, prove that $f \circ \varphi$ is a measurable function on X.

c) (10 points) Prove that for all $f \in L^1(Y, \nu)$, we have

$$\int_Y f\, d\nu = \int_X (f \circ \varphi)\, d\mu.$$

3) Suppose $E \subseteq [0, 1]$ is such that for every interval $I \subseteq [0, 1]$, $m^*(I) = m^*(I \cap E) + m^*(I \cap E^c)$. Prove E is (Lebesgue) measurable.

4) Suppose that (X, μ) and (Y, ν) are σ-finite measure spaces.

a) (4 points) Show that if h, k belong to $L^2(Y, \nu)$, then $\left| \int_Y h(y)k(y)\, d\nu(y) \right| \leq \|h\|_{L^2(Y)} \|k\|_{L^2(Y)}$.

b) (12 points) Suppose now that $K \in L^2(X \times Y, \mu \times \nu)$. If $g \in L^2(Y, \nu)$ prove that for almost every $x \in X$, $\int_Y |K(x, y)g(y)|\, d\nu(y) < \infty$.

c) (4 points) Let $f(x) = \int_Y K(x, y)g(y)\, d\nu(y)$. Prove that $f \in L^2(X, \mu)$ and that

$$\|f\|_{L^2(X)} \leq \|K\|_{L^2(X \times Y)} \|g\|_{L^2(Y)}.$$

5) Suppose (X, d) is a compact metric space, and let $C(X)$ be the collection of real-valued continuous functions on X. Recall that any compact metric space is separable (don't prove this) and let $\{x_n\}_{n=1}^{\infty}$ be a countable dense subset of X. Put $f_n(x) = d(x, x_n)$ and let ϕ be the function on X given by $\phi(x) = 1$ for every $x \in X$. Let $\mathcal{A}$ be the smallest subalgebra of $C(X)$ containing each f_n and ϕ.

a) (7 points) If $h \in C(X)$ and $\varepsilon > 0$, prove there exists $g \in \mathcal{A}$ such that for every $x \in X$, $|h(x) - g(x)| < \varepsilon$.

b) (5 points) Describe the form of a typical function in $\mathcal{A}$.

c) (8 points) For $f, g \in C(X)$, let $\Delta(f, g) := \sup_{x \in X} |f(x) - g(x)|$. Then $(C(X), \Delta)$ is a separable metric space. (The issue is the separability—no points will be awarded for proving Δ is a metric, so don't bother proving Δ is a metric.)

6) Let $F(t) = \int_{[0,\infty)} e^{-xt} \frac{\sin x}{x} \, dm(x)$.

a) (9 points) Prove that $\lim_{t\to\infty} F(t) = 0$.

b) (9 points) Prove that for $t > 0$, $F'(t) = -\frac{1}{1+t^2}$.

c) (2 points) Prove that for $t > 0$, $F(t) = \frac{\pi}{2} - \arctan t$.

7) Let $f_0 : [a,b] \to \mathbb{R}$ be an *increasing* function (so $a \le x < y \le b$ implies $f_0(x) \le f_0(y)$). Extend f_0 to a function $f : \mathbb{R} \to \mathbb{R}$ by defining

$$f(x) = \begin{cases} f_0(a) & \text{if } x < a \\ f_0(x) & \text{if } x \in [a,b] \\ f_0(b) & \text{if } x > b \end{cases}.$$

Let $g_n(x) := n(f(x + 1/n) - f(x))$ (where $n \in \mathbb{N}$).

a) (2 points) What can be said about the existence of $f'(x)$?

b) (18 points) Use Fatou's Lemma applied to the sequence g_n to show that $f' \in L^1[a,b]$ and

$$\int_a^b f' \, dm \le f(b) - f(a).$$

8) Let P be the set of all irrational numbers, viewed as a metric space with the usual Euclidian metric. For each $n \in \mathbb{N}$, let $G_n \subseteq P$ be a (relatively) open subset of P which is also dense in P. Prove that $\cap_{n=1}^{\infty} G_n$ is dense in P.