Math 901–902 Comprehensive Exam

June 2014

Instructions: Do two problems from each of the three sections, for a total of six problems.

If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

1. Part I: Group Theory and Representation Theory

(1) Let k be a field. This problem concerns the number of non-isomorphic irreducible k-linear representations of the cyclic group of order $n \geq 2$.
  (a) Find (with justification) this number when $k = \mathbb{C}$.
  (b) Prove that this number is equal to the number of positive integer divisors of n when $k = \mathbb{Q}$.

(2) Find, with justification, all non-isomorphic irreducible $\mathbb{C}$-linear representations of A_4.

(3) A *minimal normal subgroup* of a group G is a non-trivial normal subgroup N such that N does not contain any other non-trivial normal subgroups. For a prime p, an *elementary abelian p-group* is a group isomorphic to $C_p \times \cdots \times C_p$ where C_p is cyclic of order p.

Prove that every minimal normal subgroup of a finite solvable group is an elementary abelian p-group, for some prime p.

Hint: One approach is to first prove the derived subgroup of N is normal in G.

2. Field and Galois Theory

(4) Prove or disprove that $x^5 - 6x^4 + 3 \in \mathbb{Q}[x]$ is solvable by radicals. *Tip*: You may use, without proof, the fact that for any two cycle τ and five cycle σ in S_5, σ and τ generate S_5.

(5) Let ξ_n and ξ_m be n-th and m-th primitive roots of unity. Assume that $(n, m) = 1$. Prove $\mathbb{Q}(\xi_n) \cap \mathbb{Q}(\xi_m) = \mathbb{Q}$.

(6) Let p be a postive prime integer. Prove the Galois group of $x^4 - p$ over $\mathbb{Q}$ is isomorphic to D_8 (the dihedral group of order 8).

3. Rings and Modules

(7) Let R be a left semi-simple ring and let I, J be two minimal left ideals. Prove that I and J are isomorphic (as left R-modules) if and only if there is an element $r \in R$ such that $J = Ir$.

(8) Recall that the (left) Jacobson radical of a ring R, denoted by $J(R)$, is the intersection of all maximal left ideals of R. Let R be a left Artinian ring. Prove that R is semisimple if and only if $J(R) = 0$.

(9) Let $A \subseteq B$ be an integral extension of commutative rings. Assume that $a \in A$ is a unit in B. Prove that a is a unit in A.