Math 901–902 Comprehensive Exam

3rd June 2005, 1–5pm

Solve two problems from each of the three sections, for a total of *six* problems. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.
Justify all your answers.

A. **Group Theory**

1. Let G be a group and H a subgroup of G of finite index. Set $n = (G : H)$. Prove that H contains a normal subgroup K of G such that $(G : K)$ divides $n!$

2. Let p be an odd prime number and S_p the symmetric group on $\{1, \ldots, p\}$. Let $g \in S_p$ be the p-cycle $(1, \ldots, p)$.

   (a) Find the centralizer of g in S_p.

   (b) Find the orders of the centralizer of g and the normalizer of the subgroup $\langle g \rangle$.

   (c) Can the normalizer of $\langle g \rangle$ be abelian?

3. Prove that, up to isomorphism, there are exactly two groups of order $5 \times 7 \times 11$.

B. **Field and Galois Theory**

4. Find the Galois group of the splitting field over $\mathbb{Q}$ of the polynomial $x^4 - 14x^2 + 9$.

5. Given a field F let $F^\times$ denote the multiplicative group of the non-zero elements of F. For each positive integer n, set $F^{\times n} = \{x \in F^\times \mid x = y^n \text{ for some } y \in F^\times\}$.

   Prove the following statements.

   (a) $\mathbb{Q}^\times / \mathbb{Q}^{\times n}$ is infinite for each integer $n \geq 2$.

   If F is a finite field, then

   (b) $F^\times / F^{\times 2} = \{1\}$ when $\operatorname{char}(F) = 2$ and $F^\times / F^{\times 2} \cong \{\pm 1\}$ when $\operatorname{char}(F) \neq 2$.

   (c) Every element of F is a sum of two squares (one of which is possibly 0).

   Hint: For part (c): first prove it is enough to find an element which is a sum of two squares, but not a square.

6. Let a, b, c, d be integers such that

$$n = |\det(A)| \neq 0, \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   Let x and y be variables, set $L = \mathbb{C}(x, y)$ and $K = \mathbb{C}(x^a y^b, x^c y^d)$.

   (a) Prove that L is a finite extension of K and that $(L : K) = n$.

   (b) Show that L/K is Galois and find the Galois group.

Hints: Using elementary row and column operations (over $\mathbb{Z}$) one can transform the matrix A into a matrix of the form

$$\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$$

where e_1 and e_2 are non-zero integers with $e_1 \mid e_2$.

Note also that $\mathbb{C}(x^a y^b, x^c y^d) = \mathbb{C}(x^a y^b, x^{c+as} y^{d+bs})$ for each $s \in \mathbb{Z}$.

C. Representation theory

7. Let F be a field and let $f(x)$ be a polynomial over F. Set $R = F[x]/(f(x))$. Find necessary and sufficient conditions on $f(x)$ for R to be semi-simple.

8. (a) Give an example to demonstrate that an element in a ring may have a right inverse, but no left inverse.
   (b) Prove that if each non-zero element of a ring R has a right inverse, then R is a division ring.
   (c) Prove that a left Artinian ring with no zero divisors (that is to say, one in which $x \cdot y = 0$ implies either $x = 0$ or $y = 0$) is a division ring.

9. This question concerns the representations of D_4, the group of symmetries of a square, over $\mathbb{C}$.
   (a) Write down a $\mathbb{C}$-vectorspace basis for the center of the group algebra $\mathbb{C}[D_4]$.
   (b) How many irreducible representations does D_4 have, and what are their degrees?
   (c) Write down (with justifications!) the character table of D_4.