# Math 901-902 Comprehensive Exam

June 7, 2002 1–5pm

Do two of the three given problems from each of the three sections, for a total of *six* problems.
If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

## I Group Theory

I.1 Prove that every group of order $2^n \cdot 3$, for $n \geq 1$, is solvable.

I.2 Find (with proof) the smallest odd integer $n$ such that there is a non-abelian group of order $n$. (In addition to showing that every group of odd order less than $n$ is abelian, you should construct, with justification, a group of order $n$ which is non-abelian.)

I.3 Let $G$ be a *non-abelian* group of order $p^3$, for some prime $p$. Prove that every subgroup of $G$ of order $p^2$ contains the center of $G$.

## II Field Theory and Galois Theory

II.1 Let $F$ be the splitting field of the polynomial $x^4 - 7$ over $\mathbb{Q}$. By computing the Galois group of $F$ over $\mathbb{Q}$, find all intermediate fields $\mathbb{Q} \subset E \subset F$ such that $E$ is normal over $\mathbb{Q}$.

II.2 Let $F \subset K$ be a finite Galois extension and assume $E$ is an intermediate field. Let $G = Aut_F(K)$ and $H = Aut_E(K)$. (Recall that for an arbitrary field extension $L \subset L'$, the group $Aut_L(L')$ — sometimes written $Gal(L'/L)$ — consists of field automorphisms of $L'$ fixing $L$.)

(a) Prove $N_G(H)$ is equal to $\{g \in G \mid g(E) = E\}$.

(b) Prove $N_G(H)/H \cong Aut_F(E)$.

II.3 Prove that $\mathbb{C}$, the field of complex numbers, is algebraically closed. (Give an algebraic proof, using Sylow theorems and Galois theory. The only analytic tools you should need are the following facts: (a) Every real polynomial of odd degree has a real root; and (b) every complex number has a square root.)

## III Rings and Modules

III.1 Prove that a finitely generated projective module $P$ over a local ring $R$ is free.

III.2 Let $R$ be a commutative ring (with identity) and let $\mathfrak{p}$ be a minimal prime ideal of $R$. Prove that every element of $\mathfrak{p}$ is a zero-divisor of $R$. (Recall that a zero-divisor of a ring $R$ is an element $x$ such that $xy = 0$ for some non-zero element $y$.) *Hint*: Consider the multiplicative set $S = \{ab \mid a \notin \mathfrak{p}$ and $b$ is a non-zero-divisor of $R\}$.

III.3 Suppose $R$ is an integral domain. Prove that the following assertions are equivalent.

(a) $R$ is normal (i.e., $R$ is integrally closed in its field of fractions).

(b) $S^{-1}R$ is normal for every multiplicatively closed subset $S$ of $R$ (with $0 \notin S$).

(c) $R_m$ is normal for each maximal ideal $m$ of $R$.