Math 901–902 Comprehensive Exam

January 17, 2007, 2–6pm

Solve two problems from each of the three parts, for a total of *six* problems. If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Justify all your answers.

Part A.

(1) Prove that every group of order 150 is solvable.

(2) Let G be a finite group. Prove that G is cyclic if and only if G has exactly one subgroup of order d for each positive divisor d of $|G|$.

(3) Classify, up to isomorphism, all groups of order 44. Be sure to prove the groups you find are not isomorphic to each other.

Part B.

(4) Let E/F be a finite Galois field extension and K and L intermediate fields. Suppose $E = KL$ and $F = K \cap L$. Prove that if either K or L is normal over F then $[E : F] = [K : F][L : F]$.

(5) Let E/F be a normal field extension and $f(x) \in F[x]$ an irreducible polynomial. Suppose $g(x)$ and $h(x)$ are monic irreducible factors of $f(x)$ in $E[x]$. Prove that there exists an automorphism σ of E fixing F such that $g = h^\sigma$. (Here, h^σ denotes the polynomial obtained by applying σ to the coefficients of h.)

(6) Let F be a field of characteristic $p > 0$. Prove every algebraic field extension of F is separable if and only if $F = F^p$. (Here, $F^p = \{a^p \mid a \in F\}$.)

Part C.

(7) Let R be a (possibly non-commutative) ring and suppose E is a faithful and semi-simple left R-module. (Recall that "faithful" means that if, for some $r \in R$, we have $re = 0$ for all $e \in E$, then $r = 0$.)

   (a) Assume, in addition, that E is finitely generated as a left R-module and prove that R is a semi-simple ring.

   (b) Prove, by way of an example, that R need not be semi-simple if E is not assumed to be finitely generated.

(8) Let G be a finite group.

   (a) Prove $g, h \in G$ are conjugate if and only if $\chi(g) = \chi(h)$ for every irreducible (complex) character χ of G.

   (b) For $g \in G$, prove g is conjugate to g^{-1} if and only if $\chi(g)$ is a real number for every irreducible (complex) character χ of G.

(9) Find, with complete justification, the character table of D_5, the group of symmetries of a regular pentagon.