

SEMIGROUPS OF MODULES: A SURVEY

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This paper is dedicated to Carl Faith and Barbara Osofsky, who continue to inspire, enlighten and energize us, as they have for over forty years.

1. INTRODUCTION

This paper is a survey of some recent developments in the study of direct-sum decompositions of modules. The idea is to examine direct-sum behavior by looking at semigroups of isomorphism classes of modules. To be precise, let R be a ring, and let \mathcal{C} be a class of modules closed under isomorphism, finite direct sums, and direct summands. Assume there is a set $V(\mathcal{C}) \subseteq \mathcal{C}$ of representatives; thus each element $M \in \mathcal{C}$ is isomorphic to exactly one element $[M] \in V(\mathcal{C})$. We make $V(\mathcal{C})$ into an additive semigroup in the obvious way: $[M] + [N] = [M \oplus N]$. This semigroup encodes information about the way modules in \mathcal{C} decompose as direct sums of other modules in \mathcal{C} .

Given a module M , we are particularly interested in the class $+(M)$, which consists of modules that are isomorphic to direct summands of direct sums of finitely many copies of M . We will see, for example, that if R is a (commutative Noetherian) local ring and M is a finitely generated R -module, then $V(+M)$ is a finitely generated *Krull* semigroup (defined in Definition 2.4). In the terminology associated with convex polytopes [5], $V(+M)$ is a *positive normal affine* semigroup. (See definitions and various characterizations in §2.) Conversely, we have a “realization theorem”: given any finitely generated Krull semigroup H , there is a one-dimensional local domain R with a finitely generated R -module M such that $V(+M) \cong H$. These two results show exactly how badly Krull-Schmidt uniqueness can fail for direct-sum decompositions of finitely generated modules over local rings. For example, using the realization theorem, one can find an indecomposable finitely generated module M over a local ring such that $M, M^{(2)}, \dots, M^{(4)}$ have only the obvious direct-sum decompositions, whereas $M^{(5)} \cong V \oplus W$, where V and W are non-isomorphic indecomposables. On the other hand, using the fact that $V(+M)$ is a Krull semigroup, one can show, for example, that there is no local ring R with indecomposable finitely generated modules M and N such that $M^{(2)} \cong N^{(3)}$. More subtle restrictions are mentioned in Section 9.

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Similarly, $\mathsf{V}(+(M))$ is a finitely generated Krull semigroup whenever M is an Artinian right R -module (over any ring R), and again there is a realization theorem: Every finitely generated semigroup arises in this way.

In Sections 7 and 8 of the paper we discuss non-finitely generated Krull semigroups and realization theorems.

2. KRULL SEMIGROUPS

Here we recall some of the basic terminology in the theory of Krull semigroups. All of our semigroups are written additively and have a neutral element 0. Moreover, we assume always that semigroups are

- (1) *commutative* ($x + y = y + x$),
- (2) *cancellative* ($x + y = x + z \implies y = z$), and
- (3) *reduced* ($x + y = 0 \implies x = y = 0$).

Of course, all of these properties except (2) are automatically satisfied for the semigroups $\mathsf{V}(\mathcal{C})$, and we impose appropriate conditions on the rings and modules to ensure that cancellation holds.

The theory of (commutative, cancellative) semigroups, often called *monoids*, has developed along two almost disjoint paths: first, as part of a successful campaign to divorce the multiplicative theory of integral domains from the underlying additive group [8], [22], [7]; and second, in the remarkable applications of commutative algebra to problems in combinatorics and topology [34, Chapter 1], [5, §6]. The present survey concerns yet another, more recent development—the use of semigroups in module theory. The first two developments tend to use different terminology, and we shall use a blend, occasionally pointing out how concepts from one school fit into the other.

For elements x and y in a semigroup H , we say x *divides* y , and we write $x \mid y$, provided there is an element $z \in H$ such that $x + z = y$. If necessary, e.g., as in the following definition, we write $x \mid_H y$.

Definition 2.1. A semigroup homomorphism $\varphi : H \rightarrow K$ is a *divisor* homomorphism provided, for $x, y \in H$, we have $x \mid_H y \iff \varphi(x) \mid_K \varphi(y)$. If H is a subsemigroup of K , we say H is a *full* subsemigroup of K provided the inclusion map is a divisor homomorphism, that is, for all $x, y \in H$ we have $x \mid_H y \iff x \mid_K y$.

Since all of our semigroups are reduced and cancellative, divisor homomorphisms are always injective, and thus a divisor homomorphism $H \rightarrow K$ embeds H as a full subsemigroup of K .

The free semigroup 2.2. Let \mathbb{N}_0 denote the semigroup of non-negative integers. For an index set (or natural number) b , we denote by $\mathbb{N}_0^{(b)}$ the direct sum of b copies of \mathbb{N}_0 . Thus $\mathbb{N}_0^{(b)}$ is the *free semigroup with basis b* . In $\mathbb{N}_0^{(b)}$ we have $x \mid y$ if and only if $x \leq y$ relative to the coordinatewise partial ordering: If z_i denotes the i^{th} coordinate of an element $z \in \mathbb{N}_0^{(b)}$, then $x \leq y \iff x_i \leq y_i$ for all $i \in b$. Therefore, if H is a full subsemigroup of $\mathbb{N}_0^{(b)}$, we have $x \mid_H y \iff x_i \leq y_i$ for each i .

Examples 2.3. (1) The subsemigroup $H := \{[x] \in \mathbb{N}_0^{(2)} \mid x \equiv y \pmod{3}\}$ is a full subsemigroup of $\mathbb{N}_0^{(2)}$.

(2) Similarly, the subsemigroup Γ of $\mathbb{N}_0^{(3)}$ consisting of all the triples (x, y, z) satisfying $x + 4y = 5z$ is a full subsemigroup of $\mathbb{N}_0^{(3)}$.

(3) Examples 2.3.1 and 2.3.2 are instances of the following: If G is an abelian group and $\varphi : \mathbb{N}_0^{(b)} \rightarrow G$ is a semigroup homomorphism, then $\varphi^{-1}(0_G)$ is a full subsemigroup of $\mathbb{N}_0^{(b)}$.

Definition 2.4. A *Krull semigroup* is a semigroup H admitting a divisor homomorphism $H \rightarrow \mathbb{N}_0^{(b)}$ for some (possibly infinite) set b . Equivalently, H is isomorphic to a full subsemigroup of a free semigroup.

We sometimes consider a stronger condition on subsemigroups of $\mathbb{N}_0^{(b)}$:

Definition 2.5. A subsemigroup H of $\mathbb{N}_0^{(b)}$ is an *expanded subsemigroup* of $\mathbb{N}_0^{(b)}$ provided $H = \mathbb{N}_0^{(b)} \cap L$ for some \mathbb{Q} -subspace L of $\mathbb{Q}^{(b)}$.

If t is a positive integer, every \mathbb{Q} -subspace L of $\mathbb{Q}^{(t)}$ can be represented as the kernel of a matrix over \mathbb{Q} (take the matrix with a basis for L^\perp as the rows). Thus a subsemigroup H of $\mathbb{N}_0^{(t)}$ is an expanded subsemigroup of $\mathbb{N}_0^{(t)}$ if and only if H is the set of non-negative integer solutions to a finite system of homogeneous linear equations over \mathbb{Z} . Semigroups arising in this way are sometimes called *diophantine semigroups*.

We pay particular attention to *finitely generated* Krull semigroups. These turn out to be the same gadgets as the positive normal affine semigroups studied in [5]. We review the terminology here. A *positive affine semigroup* is a semigroup isomorphic to a finitely generated subsemigroup of $\mathbb{N}_0^{(t)}$ for some positive integer t . (The expression “positive semigroup” means that $x + y = 0$ only if $x = y = 0$, a property that we have tacitly attributed to all semigroups.) The positive affine semigroup H is *normal* provided, for each $x, y \in H$, if $nx \mid ny$ for some positive integer n , then $x \mid y$. Equivalently, let G denote the group of differences of H (i.e., the smallest group containing H); then H is normal if and only if, for every $g \in G$ and every positive integer n , $ng \in H$ implies that $g \in H$.

Recall that a *clutter* (or *antichain*) is a subset S of a poset such that no two elements of S are comparable. A key property of the poset $\mathbb{N}_0^{(t)}$ is that every clutter is finite, cf. [9,

Theorem 9.18]. A proof using the Hilbert Basis Theorem can be found in the first chapter of Stanley's book [34]. An alternate approach is given in the next lemma, whose proof is left to the reader:

Lemma 2.6. *Consider the following property of a partially ordered set X :*

(†) *X has the descending chain condition and every clutter in X is finite.*

If X and Y are posets satisfying (†), then $X \times Y$ (with coordinate-wise partial ordering) satisfies (†). In particular, since \mathbb{N}_0 satisfies (†), so does $\mathbb{N}_0^{(t)}$ for every positive integer t .

Proposition 2.7. *The following conditions on a semigroup H are equivalent:*

- (1) *H is a positive normal affine semigroup.*
- (2) *H is isomorphic to a full subsemigroup of $\mathbb{N}_0^{(t)}$ for some $t \geq 1$.*
- (3) *H is isomorphic to an expanded subsemigroup of $\mathbb{N}_0^{(u)}$ for some $u \geq 1$.*
- (4) *H is a finitely generated Krull semigroup.*
- (5) *$H \cong G \cap \mathbb{N}_0^{(t)}$ for some $t \geq 1$ and some subgroup G of $\mathbb{Z}^{(t)}$.*

Proof. (1) \implies (2) by [5, Exercise 6.1.10, p. 263]. If H is a full subsemigroup of $\mathbb{N}_0^{(t)}$ and G is the subgroup of $\mathbb{Z}^{(t)}$ generated by H , it is easy to see that $H = G \cap \mathbb{N}_0^{(t)}$. Therefore (2) implies (5).

Assume (5). Then H is a Krull semigroup, and, by Lemma 2.6, H is generated by its minimal non-zero elements, which are finite in number. Thus (5) implies (4). (Cf. [9, Corollary 9.19].)

To see that (4) implies (1), suppose H is a full subsemigroup of $\mathbb{N}_0^{(b)}$ and that H is finitely generated. There is a finite subset t of b such that $h \subseteq \mathbb{N}_0^{(t)}$. This shows that H is a positive affine semigroup. Since H is a full subsemigroup of $\mathbb{N}_0^{(t)}$, it follows that H is normal.

We have shown that (1), (2), (4) and (5) are equivalent. Obviously (3) implies (2). A proof that (1) implies (3), attributed to Hochster, is outlined in [5, Exercise 6.4.16, p. 290]. \square

Here is a simple example that might give the reader some indication of why (2) implies (3). The semigroup $H := \{[x] \in \mathbb{N}_0^{(2)} \mid x \equiv y \pmod{3}\}$ of (2.3.1) is *not* the restriction to $\mathbb{N}_0^{(2)}$ of the kernel of a matrix. However, H is isomorphic to $H_1 := \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{N}_0^{(3)} \mid x + 2y = 3z \right\}$.

We remark that for $t > 1$ the semigroup $\mathbb{N}_0^{(t)}$ has subsemigroups that are not finitely generated, cf. [9, Ex. 12, p. 137]. On the other hand, all finitely generated commutative semigroups are finitely presented, cf. [9, Theorem 9.28].

We now return to the examples and properties of Krull semigroups alluded to in §1.

Atoms and decompositions 2.8. Recall that an *atom* of a semigroup is a non-zero element that cannot be written as a sum of two non-zero elements. If H is a full subsemigroup of $\mathbb{N}_0^{(b)}$,

the atoms of H are just the minimal non-zero elements, with respect to the coordinatewise ordering on \mathbb{N}_0^b . For example, the atoms of the semigroup Γ of (2.3.2) are $a := (1, 1, 1)$, $b := (5, 0, 1)$ and $c := (0, 5, 4)$. For $n \leq 4$, the decomposition of na as a sum of atoms is unique, whereas $5a = b + c$.

Remark 2.9. In a Krull semigroup H one cannot have atoms x and y such that $2x = 3y$. To see this, we may assume that H is a full subsemigroup of some free semigroup $\mathbb{N}_0^{(b)}$. Such a relation would imply that $y \leq x$ in the coordinatewise ordering on $\mathbb{N}_0^{(b)}$. But then $y \mid x$ and, since x is an atom, y must equal x . Then $2x = 3x$, and cancellation implies that $x = 0$, a contradiction.

3. PROJECTIVE MODULES OVER SEMILOCAL RINGS

For folks in the non-commutative world, a ring R is *semilocal* provided $R/\text{J}(R)$ is semisimple Artinian, where $\text{J}(R)$ is the Jacobson radical. In this section we adhere to that terminology. Let us denote by $\text{mod}(R)$, respectively, $\text{pmod}(R)$ the class of finitely generated right R -modules, respectively, finitely generated projective right R -modules. The function $\text{pmod}(R) \rightarrow \text{pmod}(R/\text{J}(R))$ taking P to $P/P\text{J}(R)$ gives a semigroup homomorphism $\mathsf{V}(\text{pmod}(R)) \xrightarrow{\iota} \mathsf{V}(\text{pmod}(R/\text{J}(R)))$. (Of course $\text{pmod}(R/\text{J}(R)) = \text{mod}(R/\text{J}(R))$.) In fact, ι is a divisor homomorphism [20]. For, suppose $P, Q \in \text{pmod}R$ and $[P/P\text{J}(R)] \mid [Q/Q\text{J}(R)]$. There is then a surjective homomorphism $f : Q/Q\text{J}(R) \twoheadrightarrow P/P\text{J}(R)$. We can lift f to a homomorphism $g : Q \rightarrow P$. By Nakayama's lemma, g is surjective, and thus $[P] \mid [Q]$.

Now every finitely generated $R/\text{J}(R)$ -module is uniquely (up to isomorphism) a direct sum of simple modules, and thus $\mathsf{V}(\text{pmod}R/\text{J}(R))$ is the free semigroup generated by the finitely many isomorphism classes of minimal right ideals of R . By Proposition 2.7, $\mathsf{V}(\text{pmod}R)$ is a finitely generated Krull semigroup.

The main theorem of [16] says that every finitely generated Krull semigroup can be realized in this way. In fact, every full embedding $H \hookrightarrow \mathbb{N}_0^{(t)}$ can be realized as the map $\mathsf{V}(\text{pmod}(R)) \xrightarrow{\iota} \mathsf{V}(\text{pmod}(R/\text{J}(R)))$, as long as some element of H maps to an element $x = (x_1 \dots x_t) \in \mathbb{N}_0^{(t)}$ with each $x_i > 0$. We call such an element of $\mathbb{N}_0^{(t)}$ *strictly positive*. (Notice that $\iota([R])$ is strictly positive, since every simple module occurs as a direct summand of $R/\text{J}(R)$.) The next remark shows that we can always assume that a given subsemigroup H of $\mathbb{N}_0^{(t)}$ contains a strictly positive element. We can, if necessary, replace the embedding $H \hookrightarrow \mathbb{N}_0^{(t)}$ with the composite $H \hookrightarrow \mathbb{N}_0^{(t)} \xrightarrow{\pi} \mathbb{N}_0^{(b)}$, where b is a proper subset of $\{1, \dots, t\}$ and π is the projection.

Remark 3.1. Suppose H is a subsemigroup of $\mathbb{N}_0^{(t)}$. Then there exists a subset $b \subseteq \{1, \dots, t\}$ such that

- (1) If $(g_1, \dots, g_t) \in H$ and $i \notin b$, then $g_i = 0$; and
- (2) There is an element $(h_1, \dots, h_t) \in H$ such that $h_i > 0$ for all $i \in b$.

Proof. Choose an element $h \in H$ with the largest number of positive coordinates, and let $b = \{i \mid h_i > 0\}$. If there were an element $g \in H$ violating (1), then $g + h$ would have more positive coordinates than h , a contradiction. \square

Theorem 3.2 (Facchini and Herbera [16]). *Let H be a finitely generated Krull semigroup, and let $H \xrightarrow{\varphi} \mathbb{N}_0^{(t)}$ be a full embedding. Assume there is an element $h \in H$ such that $\varphi(h)$ is strictly positive. Let k be any field. Then there exist a semilocal, right hereditary k -algebra R and a commutative diagram of semigroups*

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & \mathbb{N}_0^{(t)} \\ \alpha \downarrow & & \beta \downarrow \\ V(\text{pmod}(R)) & \xrightarrow{\iota} & V(\text{pmod}(R/J(R))) \end{array}$$

with the following properties:

- (1) α and β are isomorphisms.
- (2) $\alpha(h) = [R]$.

The proof of this result is very intricate and uses some surprising constructions due to George Bergman [4] and a localization technique due to P. M. Cohn [10]. We sketch the proof of a similar result, Theorem 5.1, in §5. We close this section with an amusing application from [16].

Example 3.3 ([16, Example 5.8]). *Let r and s be arbitrary integers, both greater than 1. There is a semilocal, right hereditary ring R with the following properties:*

- (1) $R^{(s)}$ has, up to isomorphism, exactly two decompositions as a direct sum of indecomposable right R -modules: the obvious one, as a direct sum of s copies of R ; and as a direct sum of r pairwise non-isomorphic indecomposable modules.
- (2) For each natural number $t < s$, $R^{(t)}$ has only the obvious decomposition.

Proof. Consider the subsemigroup H of $\mathbb{N}_0^{(r)}$ generated by $h := (1, 1, \dots, 1)$, $g_1 := (s, 0, \dots, 0)$, $g_2 := (0, s, 0, \dots, 0)$, \dots , $g_r := (0, \dots, 0, s)$. One can check that H is a full subsemigroup of $\mathbb{N}_0^{(r)}$ (cf. [16, Lemma 5.5]) and that $\{h, g_1, g_2, \dots, g_r\}$ is the set of atoms of H . Notice that $sh = g_1 + \dots + g_r$ and that if $1 \leq t < s$ then th has only the obvious decomposition as a sum of atoms. Applying Theorem 3.2 to the inclusion $\varphi : H \rightarrow \mathbb{N}_0^{(r)}$, we obtain an isomorphism $H \rightarrow V(\text{pmod}(R))$ taking the strictly positive element h to $[R]$. The desired relations and non-relations in $V(\text{pmod}(R))$ then follow from those in H . \square

4. FINITELY GENERATED MODULES OVER COMMUTATIVE LOCAL RINGS

Let's switch gears completely. In this section, "local ring" means "commutative Noetherian local ring". If R is local with maximal ideal \mathfrak{m} , we let \widehat{R} denote the \mathfrak{m} -adic completion of R . Since direct-sum cancellation holds for finitely generated modules over local rings [13], we can consider the semigroup $V(\text{mod}(R))$. There is a homomorphism of semigroups $V(\text{mod}(R)) \xrightarrow{\iota} V(\text{mod}(\widehat{R}))$ taking $[M]$ to $[\widehat{M}] = [\widehat{R} \otimes_R M]$. By the Krull-Schmidt uniqueness theorem for finitely generated modules over complete local rings, $V(\text{mod}(\widehat{R}))$ is a free semigroup with basis $\{[V] \mid V \text{ is an indecomposable finitely generated } R\text{-module}\}$. The next result is an easy consequence of results in Guralnick's paper [21], but we give a different proof modeled after the proof of the same result for discrete valuation rings in [32]

Proposition 4.1. *The homomorphism $\iota : V(\text{mod}(R)) \longrightarrow V(\text{mod}(\widehat{R}))$ is a divisor homomorphism. In particular, $V(\text{mod}(R))$ is a Krull semigroup.*

Proof. (Cf. [37]) Suppose M and N are finitely generated R -modules and that $[\widehat{M}] \mid [\widehat{N}]$. Choose \widehat{R} -homomorphisms $\phi : \widehat{M} \rightarrow \widehat{N}$ and $\psi : \widehat{N} \rightarrow \widehat{M}$ such that $\psi\phi = 1_{\widehat{N}}$. Since $H := \text{Hom}_R(M, N)$ is a finitely generated R -module it follows that $\widehat{H} = \widehat{R} \otimes_R H = \text{Hom}_{\widehat{R}}(\widehat{M}, \widehat{N})$. Therefore ϕ can be approximated to any order by an element of H , that is, ϕ can be approximated modulo any given power of $\widehat{\mathfrak{m}}$. In fact, order 1 suffices: Choose $f \in \text{Hom}_R(M, N)$ such that $\widehat{f} - \phi \in \widehat{\mathfrak{m}}\widehat{H}$. Similarly, there exists $g \in \text{Hom}_R(N, M)$ with $\widehat{g} - \psi \in \widehat{\mathfrak{m}}\text{Hom}_{\widehat{R}}(\widehat{N}, \widehat{M})$. Then the image of $\widehat{g}\widehat{f} - 1_{\widehat{N}}$ is in $\widehat{\mathfrak{m}}\widehat{X}$, and now Nakayama's lemma implies that $\widehat{g}\widehat{f}$ is surjective, and therefore an isomorphism. It follows that \widehat{g} is a split surjection (with splitting map $(\widehat{g}\widehat{f})^{-1}$). By faithful flatness g is a split surjection. \square

Corollary 4.2. *Let M be a finitely generated module over a local ring R . Then $V(+M)$ is a finitely generated Krull semigroup.*

Proof. Write $\widehat{M} = V_1^{(e_1)} \oplus \cdots \oplus V_t^{(e_t)}$, where the V_i are pairwise non-isomorphic indecomposable \widehat{R} -modules and the e_i are positive integers. The homomorphism ι carries $V(+M)$ into $V(+\widehat{M})$. But $V(+\widehat{M})$ is a free semigroup, with basis $\{[V_1], \dots, [V_t]\}$. Thus $V(+M)$ is isomorphic to a full subsemigroup of $\mathbb{N}_0^{(t)}$. \square

Let $\xi : V(+M) \rightarrow V(+\widehat{M})$ be the semigroup homomorphism induced by ι . Again, we have a realization theorem (cf. [38]):

Theorem 4.3. *Let t be a positive integer, and let H be an expanded subsemigroup of $\mathbb{N}_0^{(t)}$. Assume that H contains a strictly positive element h . Then there exist a one-dimensional local domain R , a finitely generated torsion-free R -module M , and a commutative diagram*

of semigroups

$$\begin{array}{ccc} H & \xrightarrow{\subseteq} & \mathbb{N}_0^{(t)} \\ \alpha \downarrow & & \beta \downarrow \\ \mathsf{V}(+(M)) & \xrightarrow{\xi} & \mathsf{V}(+(\widehat{M})) , \end{array}$$

where

- (1) α and β are isomorphisms, and
- (2) $\alpha(h) = [M]$.

Moreover, R can be chosen to be a localization of a finitely generated \mathbb{Q} -algebra.

Sketch of the proof. The idea is to build a domain with lots of analytic branches, so that \widehat{R} has lots of minimal prime ideals, say P_1, \dots, P_s . One also arranges things so that each branch \widehat{R}/P_i has infinite Cohen-Macaulay type. A very technical construction then shows that \widehat{R} has indecomposable finitely generated torsion-free modules with arbitrary rank functions. That is, if (r_1, \dots, r_s) is any non-trivial s -tuple of non-negative integers, then there is an indecomposable finitely generated \widehat{R} -module N such that $N_{P_i} \cong \widehat{R}_{P_i}^{(r_i)}$. Then one uses the fact, due to Levy and Odenthal [30], that a finitely generated \widehat{R} -module is extended from an R -module if and only if its rank is constant.

This last fact implies that the map ξ in Theorem 4.3 above embeds $\mathsf{V}(+(M))$ as an expanded subsemigroup of $\mathsf{V}(+(\widehat{M}))$. Therefore Theorem 4.3 would be false if “expanded” were replaced by “full” in the hypotheses. In other words, we cannot use one-dimensional rings to realize a given *full* embedding. If, however, we pass to two-dimensional rings, we get a completely general realization theorem, though the ring that does the realizing is much less tractable.

Theorem 4.4. *Let t be a positive integer, and let H be a full subsemigroup of $\mathbb{N}_0^{(t)}$. Assume that H contains a strictly positive element h . Then there exist a two-dimensional local unique factorization domain R , a finitely generated reflexive R -module M , and a commutative diagram of semigroups*

$$\begin{array}{ccc} H & \xrightarrow{\subseteq} & \mathbb{N}_0^{(t)} \\ \alpha \downarrow & & \beta \downarrow \\ \mathsf{V}(+(M)) & \xrightarrow{\xi} & \mathsf{V}(+(\widehat{M})) \end{array},$$

where

- (1) α and β are isomorphisms, and
- (2) $\alpha(h) = [M]$.

Proof. Let G be the subgroup of $\mathbb{Z}^{(t)}$ generated by H , and write $\mathbb{Z}^{(t)}/G = C_1 \oplus \cdots \oplus C_s$, where each C_i is a cyclic group. Then $\mathbb{Z}^{(t)}/G$ can be embedded in $(\mathbb{R}/\mathbb{Z})^{(s)}$.

Choose any positive integer d such that $(d-1)(d-2) \geq s$. Let C be a smooth plane curve of degree d in $\mathbb{P}_{\mathbb{C}}^2$, and let A be the homogeneous coordinate ring for the embedding $C \subseteq \mathbb{P}_{\mathbb{C}}^2$. Let \mathfrak{P} be the homogeneous maximal ideal of A , and let B be the \mathfrak{P} -adic completion of the local ring $A_{\mathfrak{P}}$. Then B is a local integrally closed domain of dimension 2, and a computation (cf. [38, p. 94]) shows that the divisor class group $\mathcal{Cl}(B)$ of B contains a copy of $(\mathbb{R}/\mathbb{Z})^{(s)}$. Since $\mathbb{Z}^{(t)}/G$ embeds in $(\mathbb{R}/\mathbb{Z})^{(s)}$, there is a homomorphism $\varphi : \mathbb{Z}^{(t)} \rightarrow \mathcal{Cl}(B)$ with $\ker(\varphi) = G$. Let $\{e_1, \dots, e_t\}$ be the standard basis of $\mathbb{Z}^{(t)}$. For each $i \leq t$, write $\varphi(e_i) = [L_i]$, where L_i is a divisorial ideal of B representing the divisor class of $\varphi(e_i)$.

Next we use Heitmann's amazing theorem [25], which implies that B is the completion of some local unique factorization domain R . For each element $m = (m_1, \dots, m_t) \in \mathbb{N}_0^{(t)}$, we let $\beta(m)$ be the isomorphism class of the B -module $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$. The divisor class of this module is $m_1[L_1] + \cdots + m_t[L_t] = \varphi(m_1, \dots, m_t)$. By [36, (1.5)] (cf. also [33, Proposition 3]), the module $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$ is the completion of an R -module if and only if its divisor class is trivial, that is, if and only if $m \in G \cap \mathbb{N}_0^{(t)}$. But $m \in G \cap \mathbb{N}_0^{(t)} = H$, since H is a full subsemigroup of $\mathbb{N}_0^{(t)}$. Therefore $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$ is the completion of an R -module if and only if $m \in H$. If $m \in H$, we let $\alpha(m)$ be the isomorphism class of a module whose completion is isomorphic to $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$. In particular, by choosing a module M such that $[M] = \alpha(h)$, we get the desired commutative diagram. \square

Returning briefly to one-dimensional rings, we ask whether there is a fixed one-dimensional local ring R , with reduced completion \widehat{R} , that can realize *every* finitely generated Krull semigroup as $\mathsf{V}(+(M))$ for a suitable finitely generated R -module M . Karl Kattchee [26] has shown that the answer is no. The key idea is to look at the *embedding dimension* of a Krull semigroup H , which is the smallest t such that H can be embedded as an expanded subsemigroup of $\mathbb{N}_0^{(t)}$. By the *rank* of a semigroup H , we mean the \mathbb{Q} -dimension of $\mathbb{Q} \otimes_{\mathbb{Z}} G$, where G is the group of differences of H . Kattchee shows that, for every R -module M , the embedding dimension of $\mathsf{V}(+(M))$ is strictly less than $t + \text{rank}(+(M))$, where t is the number of minimal prime ideals of \widehat{R} . In particular, if \widehat{R} has at most t minimal prime ideals and H is a finitely generated Krull semigroup of rank 3 such that the embedding dimension of H is more than $2 + t$, then H cannot be realized as a semigroup of the form $+(M)$. He then constructs, for any n , a finitely generated Krull semigroup having rank 3 and embedding dimension at least n .

5. ARTINIAN MODULES

In a 1932 paper [29], Krull asked whether Artinian modules satisfy Krull-Schmidt uniqueness. In other words, is the decomposition of an Artinian right module into a direct sum of finitely many indecomposables necessarily unique up to isomorphism and rearrangement of the summands? The answer is “yes” if R is either commutative or right Noetherian, cf. [35]. A negative answer for general rings was finally given by Facchini, Herbera, Levy and Vámos in 1995 [17]. They showed, roughly speaking, that any failure of Krull-Schmidt demonstrated by finitely generated modules over a commutative, semilocal Noetherian ring can be replicated using Artinian modules. We use their approach, along with Theorem 4.4, to show that every finitely generated Krull semigroup is isomorphic to $\mathsf{V}(+(M))$, for a suitable Artinian module M . This result is best possible, since, as we point out in Proposition 5.2, $\mathsf{V}(+(M))$ is a Krull semigroup for any Artinian right module M over any ring.

In this section and the next, we return to the conventions of §3. In particular, a ring R is semilocal provided $R/J(R)$ is Artinian. We begin by proving another version of Theorem 3.2. The “Moreover” statement is what allows us to get results about Artinian modules.

Theorem 5.1 ([38, Theorem 4.1]). *Let H be a finitely generated Krull semigroup, and let $H \xrightarrow{\varphi} \mathbb{N}_0^{(t)}$ be a full embedding. Assume there is an element $h \in H$ such that $\varphi(h)$ is strictly positive. Then there exist a semilocal ring E and a commutative diagram of semigroups*

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & \mathbb{N}_0^{(t)} \\ \gamma \downarrow \cong & & \delta \downarrow \cong \\ \mathsf{V}(\mathrm{pmod}(E)) & \xrightarrow{\iota} & \mathsf{V}(\mathrm{pmod}(E/J(E))) \end{array}$$

with the following properties:

- (1) γ and δ are isomorphisms.
- (2) $\gamma(h) = [E]$.

Moreover, E can be taken to be a module-finite algebra over a commutative Noetherian local ring.

Proof. Choose R and M as in Theorem 4.4. Then $E := \mathrm{End}_R(M)$ is a module-finite R -algebra. There is a well-known isomorphism $\eta : \mathsf{V}(+(M)) \rightarrow \mathsf{V}(\mathrm{pmod}(E))$ taking $[N]$ to $[\mathrm{Hom}_R(M, N)]$. The inverse map takes $[P]$ to $[P \otimes_E M]$. (Cf. [12].) Let $\widehat{E} = \widehat{R} \otimes_R E = \mathrm{End}_{\widehat{R}}(\widehat{M})$. Put $J := J(E)$. Then $\widehat{J} := \widehat{R} \otimes_R J$ is the Jacobson radical of \widehat{E} , and since E/J has finite length as an R -module the natural map $E/J \rightarrow \widehat{E}/\widehat{J}$ is an isomorphism. Also, since \widehat{E} is module-finite over the complete Noetherian local ring \widehat{R} , the natural map

$\mathsf{V}(\mathrm{pmod}(\widehat{E})) \rightarrow \mathsf{V}(\mathrm{pmod}(\widehat{E}/\widehat{J}))$ is an isomorphism. The desired commutative diagram is obtained by splicing these natural isomorphisms onto the diagram provided by Theorem 4.4:

$$\begin{array}{ccc}
 H & \xrightarrow{\subseteq} & \mathbb{N}_0^{(t)} \\
 \alpha \downarrow \cong & & \beta \downarrow \cong \\
 \mathsf{V}(+(M)) & \xrightarrow{\xi} & \mathsf{V}(+(\widehat{M})) \\
 \eta \downarrow \cong & & \widehat{\eta} \downarrow \cong \\
 \mathsf{V}(\mathrm{pmod}(E)) & \longrightarrow & \mathsf{V}(\mathrm{pmod}(\widehat{E})) \\
 \downarrow & & \downarrow \cong \\
 \mathsf{V}(\mathrm{pmod}(E/J)) & \xrightarrow{\cong} & \mathsf{V}(\mathrm{pmod}(\widehat{E}/\widehat{J}))
 \end{array}$$

□

Proposition 5.2. *Let R be any ring and M an Artinian right R -module. Then $\mathsf{V}(+(M))$ is a finitely generated Krull semigroup.*

Proof. A result due to Camps and Dicks [6] says that $E := \mathrm{End}_R(M)$ is semilocal. Therefore, as we showed at the beginning of §3, $\mathsf{V}(\mathrm{pmod}(E))$ is a finitely generated Krull semigroup. The isomorphism $\eta : \mathsf{V}(+(M)) \rightarrow \mathsf{V}(\mathrm{pmod}(E))$ in the proof of Theorem 5.1 completes the proof. □

Here is the realization theorem, this time in terms of Artinian modules:

Theorem 5.3. *Let H be a finitely generated Krull semigroup. There exist a ring R and an Artinian cyclic right R -module M such that $\mathsf{V}(+(M)) \cong H$.*

Proof. By Theorem 5.1 there exists a module-finite algebra E , over a commutative Noetherian local ring, such that $H \cong \mathsf{V}(\mathrm{pmod}(E))$. By [17, Corollary 1.3], there exist a ring R and an Artinian cyclic right R -module M such that $E \cong \mathrm{End}_R(M)$. Once again, the isomorphism $\eta : \mathsf{V}(+(M)) \rightarrow \mathsf{V}(\mathrm{pmod}(E))$ completes the proof. □

6. TORSION-FREE ABELIAN GROUPS

The complicated direct-sum behavior of finite-rank torsion-free abelian groups has received a lot of attention (cf. [1] for an excellent survey). Even the torsion-free $\mathbb{Z}_{(p)}$ -modules of finite rank have a very rich structure with respect to direct-sum decompositions (although they *do* satisfy direct-sum cancellation, [1, (8.1), (8.4), (8.8)]). A realization theorem, published by Yakovlev in 2000, illustrates this richness:

Theorem 6.1 ([39]). *Let H be a finitely generated Krull semigroup and let p be a prime number. Then there is a finite-rank torsion-free $\mathbb{Z}_{(p)}$ -module G such that $V(+G)) \cong H$. \square*

Of course the module G in Theorem 6.1 is not finitely generated!

7. MODULES WITH SEMILOCAL ENDOMORPHISM RINGS

In this section we give a realization theorem for Krull semigroups that are not necessarily finitely generated. We begin with a theorem due to A. Facchini that tells us what to “realize”:

Theorem 7.1 ([14]). *Let R be a ring and \mathcal{C} a class of modules closed under isomorphism, finite direct sums, and direct summands. Assume there is a set $V(\mathcal{C}) \subseteq \mathcal{C}$ of representatives and that $\text{End}_R(M)$ is semilocal for each $M \in \mathcal{C}$. Then $V(\mathcal{C})$ is a Krull semigroup.*

The realization theorem below is due to Facchini and R. Wiegand. Perhaps surprisingly, the modules that realize a given Krull semigroup can be taken to be finitely generated and projective. Given a ring R , let $\mathcal{S}(R)$ be the class of finitely generated projective right R -modules whose endomorphism rings are semilocal.

Theorem 7.2 ([19]). *Let H be a Krull semigroup, and let $\varphi : H \rightarrow \mathbb{N}_0^{(b)}$ be a divisor homomorphism from H to a free semigroup. Let k be any field. Then there exist a k -algebra R and a commutative diagram of semigroups*

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & \mathbb{N}_0^{(b)} \\ \alpha \downarrow \cong & & \beta \downarrow \cong \\ V(\mathcal{S}(R)) & \xrightarrow{\tau} & V(\mathcal{S}(R/J(R))) \end{array},$$

in which α and β are isomorphisms and τ is the natural map induced by change of rings.

The proof is extremely technical and, like Theorem 3.2, makes heavy use of Bergman’s constructions [4] and Cohn’s inversive localization [10]. The rings that are constructed are frightfully non-commutative. Indeed, this is unavoidable, in view of the following observation due to Facchini:

Theorem 7.3 ([15, Theorem 4.2]). *Let R be a commutative ring. If $V(\text{pmod}(R))$ is a Krull semigroup, then it is a free semigroup.*

8. THE SEMIGROUP OF all FINITELY GENERATED MODULES

In this section the word “local” means “commutative local and Noetherian”. Suppose (R, \mathfrak{m}) is a one-dimensional local ring. We let $\mathcal{F}(R)$ be the class of finitely generated modules

that are free on the punctured spectrum. Thus $M \in \mathcal{F}(R)$ if and only if M_P is a free R_P -module for each minimal prime ideal P . Of course, if R is reduced then $\mathcal{F}(R) = \text{mod}(R)$. In this section we give complete sets of invariants for the semigroups $V(\mathcal{F}(R))$, where R ranges over one-dimensional local rings. The results we discuss are due to Facchini, Hassler, Karr, Klingler and R. Wiegand and appear in two papers, [18] (the Cohen-Macaulay case) and [24] (the general case).

We recall that a local ring (R, \mathfrak{m}, k) is *Dedekind-like* provided R is one-dimensional and reduced, the integral closure \bar{R} of R in its total quotient ring is generated by at most 2 elements as an R -module, and \mathfrak{m} is the Jacobson radical of \bar{R} . In [27] and [28], Klingler and Levy show, essentially, that Dedekind-like rings and their homomorphic images are exactly the Noetherian local rings of positive dimension whose finite-length modules have tame representation type. Our description of the semigroup $V(\text{mod}(R))$ for a Dedekind-like ring R depends crucially on the description of all finitely generated modules in [28]. Although the description in [28] does not include the special situation when $\hat{R}/\hat{\mathfrak{m}}$ is a purely inseparable field of degree 2 over R/\mathfrak{m} , this difficulty has no bearing on our computations. We outline the procedure in this case below the statement of Theorem 8.1.

Fix a positive integer q and an infinite cardinal τ . We put $\mathcal{H}(0, \tau) := \mathbb{N}_0^{(\tau)}$, the free semigroup of rank τ . If $q > 0$, let B be any $q \times \tau$ integer matrix such that each element of $\mathbb{Z}^{(q)}$ occurs τ times as a column of B . We let

$$\mathcal{H}(q, \tau) := \mathbb{N}_0^{(\tau)} \cap \ker(B : \mathbb{Z}^{(\tau)} \rightarrow \mathbb{Z}^{(q)}).$$

Not surprisingly, the isomorphism class of the semigroup $\mathcal{H}(q, \tau)$ does not depend on how the columns are arranged, as long as each column is repeated τ times, cf. [18, Lemmas 1.1 and 2.1]. The semigroups $\mathcal{H}(q, \tau)$ show up in the form $V(\mathcal{F}(R))$ if R is non Dedekind-like. For some Dedekind-like rings, we obtain a different semigroup: Let E be the $1 \times \mathbb{N}_0$ matrix $\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & \dots \end{bmatrix}$, and put

$$\mathcal{H}_1 := \mathbb{N}_0^{\mathbb{N}_0} \cap \ker(E : \mathbb{Z}^{(\mathbb{N}_0)} \rightarrow \mathbb{Z})$$

For a one-dimensional local ring (R, \mathfrak{m}, k) , we define the *splitting number* $\text{spl}(R)$ to be the difference $|\text{Spec}(\hat{R})| - |\text{Spec}(R)|$. Thus, for example, $\text{spl}(R) = 0$ means that the natural map $\text{Spec}(\hat{R}) \rightarrow \text{Spec}(R)$ is bijective. It is interesting that the divisor class group $C\ell(V(\mathcal{F}(R)))$ depends only on the splitting number of R . (See, for example, [8] or [22] for the definitions and basic properties of the divisor class group of a Krull semigroup. It measures in some sense the lack of unique factorization. For example, a Krull semigroup is free if and only if it has unique factorization, if and only if its divisor class group is trivial.)

Theorem 8.1 ([18, Theorem 2.2], [24, Theorem 6.3]). *Suppose (R, \mathfrak{m}, k) is a one-dimensional commutative Noetherian local ring. Let $q := \text{spl}(R)$ be the splitting number of R , and let $\tau = \tau(R) = |k| \cdot \aleph_0$.*

- (1) *If R is not a homomorphic image of a Dedekind-like ring, then $\mathsf{V}(\mathcal{F}(R)) \cong \mathcal{H}(q, \tau)$.*
- (2) *If R is a discrete valuation ring, then $\mathsf{V}(\mathcal{F}(R)) = \mathsf{V}(\text{mod}(R)) \cong \mathbb{N}_0^{(\aleph_0)}$.*
- (3) *If R is a homomorphic image of a Dedekind-like ring but is not a discrete valuation ring, and if $q = 0$, then $\mathsf{V}(\mathcal{F}(R)) = \mathsf{V}(\text{mod}(R)) \cong \mathbb{N}_0^{(\tau)}$.*
- (4) *If R is a homomorphic image of a Dedekind-like ring and $q > 0$, then $q = 1$ and $\mathsf{V}(\mathcal{F}(R)) = \mathsf{V}(\text{mod}(R)) \cong \mathbb{N}_0^{(\tau)} \oplus \mathcal{H}_1$.*

In every case, $\mathcal{C}\ell(\mathsf{V}(\mathcal{F}(R))) \cong \mathbb{Z}^{(q)}$.

In the special case where $\widehat{R}/\widehat{\mathfrak{m}}$ is purely inseparable over R/\mathfrak{m} , the completion \widehat{R} is a domain, and consequently $\mathsf{V}(\text{mod}(R)) \cong \mathsf{V}(\text{mod}(\widehat{R}))$. Even though the R -modules have so far defied classification, one can show in this case that there are τ pairwise non-isomorphic indecomposables in this case, and hence that $\mathsf{V}(\text{mod}(R)) \cong \mathbb{N}_0^{(\tau)}$.

It would be nice to have a theorem describing the semigroup of maximal Cohen-Macaulay (equivalently, finitely generated and torsion-free) modules over a one-dimensional reduced ring. The reason we do not yet have such a theorem is that we do not yet know exactly which ranks can occur for maximal Cohen-Macaulay modules over a complete, reduced one-dimensional local ring. If we know that each analytic branch has infinite Cohen-Macaulay type, however, we can appeal to the following result, proved by Andrew Crabbe:

Theorem 8.2 (Crabbe, [11]). *Let (R, \mathfrak{m}, k) be a reduced, one-dimensional, complete local ring with minimal prime ideals P_1, \dots, P_s . Assume that R/P_i has infinite Cohen-Macaulay type. Let (r_1, \dots, r_s) be an arbitrary non-trivial s -tuple of non-negative integers, and assume that $r_1 \geq r_i$ for $i = 2, \dots, s$. Put $\tau = \tau(R) := |k| \cdot \aleph_0$. Then there is a family of τ pairwise non-isomorphic indecomposable maximal Cohen-Macaulay R -modules M such that $M_{P_i} \cong R_{P_i}^{(r_i)}$ for each $i = 1, \dots, s$.*

Now let $\mathcal{CM}(R)$ denote the class of maximal Cohen-Macaulay R -modules, together with the zero-module. Using Crabbe's result, we can make minor modifications in the proof of [18, Theorem 2.2] to obtain the following description of the semigroup $\mathsf{V}(\mathcal{CM}(R))$:

Corollary 8.3. *Let (R, \mathfrak{m}, k) be a one-dimensional local ring whose completion \widehat{R} is reduced. Assume that R/P has infinite Cohen-Macaulay type for each minimal prime ideal P of R . Let $q = \text{spl}(R)$, and put $\tau = \tau(R) := |k| \cdot \aleph_0$. Then $\mathsf{V}(\mathcal{CM}(R)) \cong \mathcal{H}(q, \tau)$.*

At the other extreme, when R has finite Cohen-Macaulay type, Nick Baeth [2] has given a detailed description of the semigroups $\mathcal{CM}(R)$, in the case where R contains a field of

characteristic different from 2, 3, 5 and the residue field k is perfect. He and Melissa Luckas have worked out most of his computations in the non-equicharacteristic, non-perfect case, without assumptions on the characteristic of the residue field [3].

9. THE DIVISOR SEQUENCE OF A KRULL SEMIGROUP

Given a non-zero element x in a Krull semigroup H , we let $a_n = a_n(x)$ be the number of distinct atoms of H that divide nx . We are interested in the behavior of the *divisor sequence* $(a_1(x), a_2(x), a_3(x), \dots)$ associated to x . For example, for the element a in the example of (2.8), the sequence is $(1, 1, 1, 1, 3, 3, 3, 3, 3, \dots)$. For an arbitrary element x in a Krull semigroup, the sequence satisfies the following three properties:

- (1) $a_i \leq a_{i+1}$ for all n (non-decreasing).
- (2) There is an N such that $a_n = a_N$ for $n \geq N$ (eventually constant).
- (3) If $a_n = 1$, then $a_{n+1} \neq 2$.

To verify (3), suppose nx is divisible by only one atom, say y , and that $(n+1)x$ is divisible by exactly two atoms y and z . Since x is a sum of atoms, and thus is divisible by *some atom*, and since every divisor of x is also a divisor of nx , we see that x must be divisible only by y . Thus $x = ry$ for some $r > 0$. With $s = (n+1)r$, we have $sy = (n+1)x = ty + uz$, with $u > 0$. After cancelling some y 's, we get $vy = uz$, with $v > 0$. An argument similar to that of Remark 2.9 forces $y = z$.

At this point it is not known whether every sequence satisfying these three properties actually arises as the divisor sequence associated to some element of a Krull semigroup. If x is an atom, however, some restrictions apply, in addition to the obvious one that $a_1(x) = 1$. In fact, Hassler [23] has shown that the sequence $(1, 3, 3, 4, 4, 4, 4, \dots)$ cannot occur for an atom. Terri Moore [31] has recently obtained several additional restrictions, in the case where the element x is an atom. Conversely, she has shown that any given non-decreasing, eventually constant sequence (a_1, a_2, a_3, \dots) with $a_1 = 1$ and with no difference $a_{n+1} - a_n$ equal to 1, actually occurs as the divisor sequence of an atom in a suitable Krull semigroup.

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