BIG INDECOMPOSABLE MIXED MODULES OVER HYPERSURFACE SINGULARITIES

Wolfgang Hassler and Roger Wiegand

April 21, 2005

This paper is dedicated to Ed Enochs, whose mastery of all things homological inspires us all.

0. Introduction

This research began as an effort to determine exactly which one-dimensional local rings have indecomposable finitely generated modules of arbitrarily large constant rank. The approach, which uses a new construction of indecomposable modules via the bimodule structure on certain Ext groups, turned out to be effective mainly for hypersurface singularities. The argument was eventually replaced by a direct, computational approach [HKKW], which applies to all one-dimensional Cohen-Macaulay local rings.

In this paper we resurrect the Ext argument to build indecomposable modules of large rank over hypersurface singularities of any dimension $d \geq 1$. The main point of the construction is that, modulo an indecomposable finite-length part, the modules constructed are maximal Cohen-Macaulay modules. Thus, even when there are no indecomposable maximal Cohen-Macaulay modules of large rank, we can build short exact sequences

$$0 \to T \to X \to F \to 0,$$

in which T and X are indecomposable, T has finite length, and F is maximal Cohen-Macaulay of arbitrarily large constant rank. The main result (Theorem 2.3) on building indecomposables is quite general, and it is likely that there are other contexts where it will prove useful.

In order to state our main application, we establish some terminology. Let k be a field. By a *hypersurface singularity* we mean a commutative Noetherian local ring $(R, \mathfrak{m}, k)$ whose $\mathfrak{m}$-adic completion $\widehat{R}$ is isomorphic to $S/(f)$, where $(S, \mathfrak{n}, k)$ is a complete regular local ring and f is a non-zero element of $\mathfrak{n}^2$. A Noetherian local ring $(R, \mathfrak{m}, k)$ is *Dedekind-like* [KL1, Definition 2.5] provided R is one-dimensional and reduced, the integral closure $\overline{R}$ is generated by at most 2 elements as an R-module, and $\mathfrak{m}$ is the Jacobson radical of $\overline{R}$. (Examples include discrete valuation rings and rings such as $k[[x,y]]/(xy)$ and $\mathbb{R}[[x,y]]/(x^2+y^2)$.) If $(R, \mathfrak{m}, k)$

Hassler's research was supported by the *Fonds zur Förderung der wissenschaftlichen Forschung*, project number P16770-N12. Wiegand's was partially supported by grants from the National Science Foundation and the National Security Agency.

is a complete hypersurface singularity containing a field, we will call R an *(A_1)-singularity* provided R is isomorphic to a ring of the form

$$(\dagger) \qquad k[[x_0,\dots,x_d]]/(g+v_1x_2^2+v_1v_2x_3^2+\dots+v_1v_2v_3\cdot\ldots\cdot v_{d-1}x_d^2),$$

where each v_i is a unit of $k[[x_0,\dots,x_i]]$, $g \in k[[x_0,x_1]]$ and $k[[x_0,x_1]]/(g)$ is Dedekind-like (but not a discrete valuation ring). By adjusting g and multiplying the defining equation by v_1^{-1}, we could eliminate the unit v_1. However, the form ($\dagger$) is more convenient notationally and in fact will be essential in Corollary 5.5. If k is algebraically closed and of characteristic different from 2, we can make the change of variables $\sqrt{v_1\cdot\ldots\cdot v_{i-1}}x_i \mapsto x_i$ $(i=2,\dots,d)$ and put g in the form $x_0^2+x_1^2$, so that R acquires the more palatable form $k[[x_0,\dots,x_d]]/(x_0^2+x_1^2+\dots+x_d^2)$.

We consider the following property of a commutative Noetherian local ring $(R,\mathfrak{m},k)$:

($\ddagger$) For every positive integer m, there exist an integer $n \ge m$ and an indecomposable maximal Cohen-Macaulay R-module F such that $F_P \cong R_P^{(n)}$ for every prime ideal $P \ne \mathfrak{m}$.

At the opposite extreme, we say that a Gorenstein local ring $(R,\mathfrak{m},k)$ has *bounded Cohen-Macaulay type* provided there is a bound on the number of generators required for indecomposable maximal Cohen-Macaulay R-modules. (We restrict to Gorenstein rings to avoid any possible conflict with the terminology of [LW2]. Cf. [LW2, Lemma 1.4].) In our context, at least in the complete case, there is a dichotomy, the proof of which will be deferred to §3:

Theorem 0.1. *Let $(R,\mathfrak{m},k)$ be a hypersurface singularity of positive dimension, containing a field of characteristic different from 2. If $\widehat{R}$ does not have bounded Cohen-Macaulay type, then both R and $\widehat{R}$ satisfy ($\ddagger$).*

The rings of bounded Cohen-Macaulay type of course include those of *finite* Cohen-Macaulay type (those having only finitely many indecomposable maximal Cohen-Macaulay modules up to isomorphism). Among excellent Gorenstein rings containing a field, the rings of finite Cohen-Macaulay type have been classified completely (cf. [LW1, §0]). It turns out ([LW2] and Corollary 5.5 below) that if $(R,\mathfrak{m},k)$ is a complete hypersurface singularity containing a field of characteristic different from 2, then R has bounded but infinite Cohen-Macaulay type if and only if R is either an (A_∞)- or (D_∞)-singularity, that is, R is isomorphic to a ring as in ($\dagger$) but with $g=$ either x_1^2 or $x_0x_1^2$.

We now state our main application of Theorem 2.3. The proof will be given in §5.

Theorem 0.2. *Let $(R,\mathfrak{m},k)$ be a hypersurface singularity of positive dimension, containing a field of characteristic different from 2. Assume that the $\mathfrak{m}$-adic completion $\widehat{R}$ has bounded Cohen-Macaulay type but is not an (A_1)-singularity. Given any positive integer m, there exist an integer $n \ge m$ and a short exact sequence of finitely generated R-modules*

$$(0.2.1) \qquad 0 \to T \to X \to F \to 0,$$

in which

(a) *T is an indecomposable finite-length module,*

(b) X *is indecomposable,*
(c) F *is maximal Cohen-Macaulay, and*
(d) $F_P \cong X_P \cong R_P^{(n)}$ *for every prime ideal* $P \neq \mathfrak{m}$.

Putting Theorems 0.1 and 0.2 together, we have the following:

Corollary 0.3. *Let $(R, \mathfrak{m}, k)$ be a hypersurface singularity of positive dimension, containing a field of characteristic different from 2. Assume $\widehat{R}$ is not an (A_1)-singularity. Given any integer m, there exist an integer $n \geq m$, an indecomposable finitely generated R-module X, and a finite-length submodule $T \subsetneq X$ (possibly $T = 0$) such that X/T is maximal Cohen-Macaulay and $X_P \cong R_P^{(n)}$ for every prime ideal $P \neq \mathfrak{m}$.* □

We have been unable to determine whether or not the conclusion of Corollary 0.3 holds if R is an (A_1)-singularity, but we expect that it always fails. More precisely, we conjecture that if R is an (A_1)-singularity then there is a bound b, depending only on $\dim(R)$, such that for every short exact sequence (0.2.1) satisfying (a) – (c) and every non-maximal prime ideal, X_P is a free R_P-module of rank at most b. This is true in dimension one [KL2], where one can take $b = 2$.

Here is a brief outline of the paper: In §1 and §2 we establish our main result, Theorem 2.3, on building indecomposable modules. In §3 we review some known results on syzygies and double branched covers, and we prove Theorem 0.1. In §4 we work through some details of a construction of large indecomposable finite-length modules, and in §5 we assemble the results of §2 – §4 to prove Theorem 0.2.

1. Bimodules

In this section let R be a commutative Noetherian ring, and let A and B be module-finite R-algebras (not necessarily commutative). Let ${}_AE_B$ be an $A-B$-bimodule. We assume E is *R-symmetric*, that is, $re = er$ for $r \in R$ and $e \in E$. Furthermore we assume that E is module-finite over R. The Jacobson radical of a (not necessarily commutative) ring C is denoted by $\mathrm{J}(C)$, and the ring C is said to be *local* provided $C/\mathrm{J}(C)$ is a division ring, equivalently [F, Proposition 1.10], the set of non-units of C is closed under addition. (The emergence of local rings in this non-commutative sense has forced the annoying repetition of "commutative Noetherian local ring" where most commutative people would say simply "local ring".) The following lemma assembles some useful trivialities that allow us to transfer ring properties across the bimodule E.

Lemma 1.1. *Let $\alpha : {}_AA \longrightarrow {}_AE$ and $\beta : B_B \longrightarrow E_B$ be module homomorphisms, and assume that $\alpha(1_A) = \beta(1_B)$. Put $C := \beta^{-1}(\alpha(A))$.*

(1) *If $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $\alpha(a_i) = \beta(b_i), i = 1, 2,$ then $\alpha(a_1a_2) = \beta(b_1b_2)$.*
(2) *C is an R-subalgebra of B.*
(3) $\mathrm{Ker}(\beta) \cap C$ *is an ideal of C; thus $D := \beta(C)$ has a unique ring structure making $\beta' : C \twoheadrightarrow D$ (the map induced by β) a ring homomorphism.*
(4) *Assume $\alpha(A) \subseteq \beta(B)$. Then the map $\alpha' : A \twoheadrightarrow D$ induced by α is a ring homomorphism (where D has the ring structure of* (3)*).*

Proof. (1) We have $\alpha(a_1a_2) = a_1\alpha(a_2) = a_1\beta(b_2) = a_1\beta(1_Bb_2) = a_1\beta(1_B)b_2 = a_1\alpha(1_A)b_2 = \alpha(a_11_A)b_2 = \alpha(a_1)b_2 = \beta(b_1)b_2 = \beta(b_1b_2)$. This proves (1), and it follows that C is a subring of B. A similar argument, using the fact that E is R-symmetric, shows that $1_Br \in C$ for each $r \in R$. Thus C is an R-subalgebra of B.

For (3), let $b_1, b_2 \in C$, with $b_2 \in \operatorname{Ker}(\beta)$. Choosing $a_1, a_2 \in A$ as in (1), we have $\beta(b_1b_2) = \alpha(a_1a_2) = a_1\alpha(a_2) = a_1\beta(b_2) = 0$. Since $\operatorname{Ker}(\beta) \cap C$ is clearly a right ideal of C, it is an ideal. To prove (4), let $a_1, a_2 \in A$, and choose $b_1, b_2 \in B$ as in (1). Then $\alpha(a_1a_2) = \beta(b_1b_2) = \beta(b_1)\beta(b_2) = \alpha(a_1)\alpha(a_2)$. $\square$

Theorem 1.2. *With notation of Lemma 1.1, assume $\alpha(1_A) = \beta(1_B)$ and $\operatorname{Ker}(\beta) \subseteq \mathrm{J}(B)$. If A is local and $\alpha(1_A) \neq 0$, then C is local.*

Proof. Suppose first that $\alpha(A) \subseteq \beta(B)$. With D as in the lemma, we have surjective ring homomorphisms

$$A \overset{\alpha'}{\twoheadrightarrow} D \overset{\beta'}{\twoheadleftarrow} C.$$

Therefore D is a (non-trivial) local ring, and to show that C is local, it will suffice to show that $\operatorname{Ker}(\beta') \subseteq \mathrm{J}(C)$. Since $\operatorname{Ker}(\beta) \subseteq \mathrm{J}(B)$, it is enough to show that $\mathrm{J}(B) \cap C \subseteq \mathrm{J}(C)$. As B is a module-finite R-algebra, left invertibility and right-invertibility are the same in B (thus we simply use the word "invertible"). Suppose now that $x \in \mathrm{J}(B) \cap C$. To show that $x \in \mathrm{J}(C)$ we must show that $z := 1 + yx$ is invertible in C for each $y \in C$. Since z *is* invertible in B, write $bz = 1$, with $b \in B$. Since B is module-finite over R, b is integral over R, say, $b^n + r_1b^{n-1} + \cdots + r_{n-1}b + r_n = 0$, with $r_i \in R$. Multiplying this equation by z^{n-1}, we see that $b \in C$, as desired.

For the general case, put $G = \alpha^{-1}(\beta(B))$. By (2) of Lemma 1.1 (with the roles of A and B interchanged), G is an R-subalgebra of A. To see that C is local, it will suffice to show that every non-unit of G is a non-unit of A. Since A is integral over R, the argument in the preceding paragraph does the job. $\square$

2. Extensions

Here we establish a context for Theorem 1.2. Let R be a commutative Noetherian ring, and let T and F be finitely generated R-modules. Put $A := \operatorname{End}_R(T)$ and $B := \operatorname{End}_R(F)$. Note that each of the R-modules $\operatorname{Ext}^n_R(F,T)$ has a natural $A - B$-bimodule structure. Indeed, any $f \in B$ induces an R-module homomorphism $f^* : \operatorname{Ext}^n_R(F,T) \to \operatorname{Ext}^n_R(F,T)$. For $x \in \operatorname{Ext}^n_R(F,T)$ put $x \cdot f = f^*(x)$. The left A-module structure is defined similarly, and the fact that $\operatorname{Ext}^n_R(F,T)$ is a bimodule follows from the fact that $\operatorname{Ext}^n(\_\,,\_)$ is an additive bifunctor. Note that $\operatorname{Ext}^n_R(F,T)$ is R-symmetric, since, for $r \in R$, multiplications by r on F and on T induce the same endomorphism of $\operatorname{Ext}^n_R(F,T)$.

Put $E = \operatorname{Ext}^1_R(F,T)$, regarded as equivalence classes of short exact sequences $0 \to T \to X \to F \to 0$. Let $\alpha : {}_AA \to {}_AE$ and $\beta : B_B \to E_B$ be module homomorphisms satisfying $\alpha(1_A) = \beta(1_B) =: [\xi]$. Then α and β are, up to signs, the connecting homomorphisms in the long exact sequences of Ext obtained by applying $\operatorname{Hom}_R(\_\_\,,T)$ and $\operatorname{Hom}_R(F,\_\_\,)$, respectively, to the equivalence class $[\xi]$ of the short exact sequence ξ. (When one computes Ext via resolutions one must adorn maps with appropriate $\pm$ signs, in order to ensure naturally of the connecting homomorphisms. In what follows, the choice of sign will not be important.)

Recall that T is a *torsion* module provided it is killed by some non-zerodivisor of R, and that F is *torsion-free* provided every non-zerodivisor of R is a non-zerodivisor on F.

Lemma 2.1. *Let R be a commutative Noetherian ring, T a finitely generated torsion module, and F a finitely generated torsion-free module. Let A, B, E be as above, and let $\alpha : {}_A A \to {}_A E$ and $\beta : B_B \to E_B$ be module homomorphisms with $\alpha(1_A) = [\xi] = \beta(1_B)$, where ξ is the short exact sequence*

$$(\xi) \qquad 0 \to T \xrightarrow{i} X \xrightarrow{\pi} F \to 0.$$

Let $\rho : \mathrm{End}_R(X) \to \mathrm{End}_R(F) =: B$ be the canonical homomorphism (reduction modulo torsion). Then the image of ρ is exactly the ring $C := \beta^{-1}\alpha(A) \subseteq B$.

Proof. By applying various Hom functors to ξ, we obtain the following diagram of exact sequences:

$$\begin{tikzcd}
 & \mathrm{Hom}_R(F,X) \arrow[r] \arrow[d] & \mathrm{Hom}_R(X,X) \arrow[d, "\pi_*"] & \\
0 \arrow[r] & B \arrow[r, "\chi", "\cong"'] \arrow[d, "\beta"'] & \mathrm{Hom}_R(X,F) \arrow[r] \arrow[d, "i_*"'] & 0 \\
A \arrow[r, "\alpha"] & E \arrow[r, "\pi^*"] & \mathrm{Ext}^1_R(X,T) &
\end{tikzcd}$$

The top square commutes, and the bottom square commutes up to sign. Clearly $\rho = \chi^{-1}\pi_*$, and an easy diagram chase shows that the image of $\chi^{-1}\pi_*$ is C. $\square$

Lemma 2.2. *Keep the notation and hypotheses of Lemma 2.1. Suppose C has no idempotents other than 0 and 1. If $X = U \oplus V$ (a decomposition as R-modules), then either U or V is a torsion module.*

Proof. Suppose $X = U \oplus V$, with both U and V non-zero, and let $f \in \mathrm{End}_R(X)$ be the projection on U (relative to the decomposition $X = U \oplus V$). Then π induces an isomorphism $\overline{\pi} : U/U_{\mathrm{tors}} \oplus V/V_{\mathrm{tors}} \to F$, and $\rho(f) \in \mathrm{End}_R(F)$ is the projection on $\overline{\pi}(U/U_{\mathrm{tors}})$. If U/U_{tors} and V/V_{tors} were both non-zero, $\rho(f)$ would be a non-trivial idempotent of C, contradiction. $\square$

The next theorem is our main result on construction of indecomposable modules.

Theorem 2.3. *Let T be a finitely generated torsion module and F a finitely generated torsion-free module over a commutative Noetherian ring R. Assume $A := \mathrm{End}_R(T)$ is local, put $B = \mathrm{End}_R(F)$, and assume that there is a right B-module homomorphism $\beta : B \to \mathrm{Ext}^1_R(F,T)$ with $\mathrm{Ker}(\beta) \subseteq \mathrm{J}(B)$. In the resulting short exact sequence*

$$(\xi) \qquad 0 \to T \to X \to F \to 0,$$

where $\beta(1_B) = [\xi] \in \operatorname{Ext}^1_R(F, T)$*, the module* X *is indecomposable.*

Proof. Let $\alpha : A \to \operatorname{Ext}^1_R(F,T)$ be the left A-module homomorphism taking 1_A to $[\xi]$. Since T is indecomposable (as its endomorphism ring is local) we may assume that $F \neq 0$. Then $\alpha(1_A) = \beta(1_B) \neq 0$. Now Theorem 1.2 implies that C is local. Suppose now that $X = U \oplus V$ with U and V non-zero. By Lemma 2.2 either U or V is torsion, say, $U \subseteq T$. Then U is a direct summand of T, whence $U = T$. But then the short exact sequence ξ splits, contradicting $\alpha(1_A) \neq 0$. □

The modules T and F in the theorem could be replaced by a torsion and torsion-free module with respect to any torsion theory for finitely generated R-modules. For example, one could take T to be any non-zero finite-length module and F a module of positive depth. The key property we need is that $\operatorname{Hom}_R(T, F) = 0$, to ensure, in Lemma 2.1, that T is a fully invariant submodule of X and that the map χ in the proof is surjective.

For lack of a convenient reference, we record the following result:

Lemma 2.4. *Let* M *be a finitely generated module over a commutative Noetherian local ring* $(R, \mathfrak{m})$*, let* Γ *be an* R*-subalgebra of* $\operatorname{End}_R(M)$ *and let* $g \in \Gamma$*. If* $g(M) \subseteq \mathfrak{m}M$*, then* $g \in \operatorname{J}(\Gamma)$*.*

Proof. It will suffice to show that $1 + hg$ is a unit of Γ for every $h \in \Gamma$. For each $x \in M$ we have $x = (1 + hg)(x) - h(g(x)) \in (1 + hg)(M) + \mathfrak{m}M$. By Nakayama's lemma, $1 + hg$ is surjective and therefore (as M is Noetherian) an automorphism. The inverse (in $\operatorname{End}_R(M)$) of $1 + hg$ is integral over R and therefore is in $R[1 + hg] \subseteq \Gamma$. □

3. Syzygies and double branched covers

We begin by assembling some known results from the literature. In this section "local ring" always means "commutative Noetherian local ring".

Let $(R, \mathfrak{m}, k)$ be local ring. Given a finitely generated R-module M, we denote by $\operatorname{syz}^n_R(M)$ the n^{th} syzygy of M with respect to a minimal free resolution of M. If we write $\operatorname{syz}^n_R(M) = F \oplus R^{(a)}$, where F has no non-zero free summand, then the module F is called the n^{th} *reduced* syzygy of M and is denoted by $\operatorname{redsyz}^n_R(M)$. Both $\operatorname{syz}^n_R(M)$ and $\operatorname{redsyz}^n_R(M)$ are well-defined up to isomorphism. Moreover, if $0 \to G \oplus R^{(b)} \to R^{(b_{n-1})} \to \cdots \to R^{(b_0)} \to M \to 0$ is exact (not necessarily minimal) and G has no non-zero free summand, then $G \cong \operatorname{redsyz}^n_R(M)$. These observations follow easily from Schanuel's lemma [M, §19, Lemma 3] and direct-sum cancellation over local rings [E]. We denote by $\mu_R(M)$ the number of generators required for the R-module M.

Lemma 3.1. *Let* $(S, \mathfrak{n}, k)$ *be a local ring, let* z *be a non-zerodivisor in* $\mathfrak{n}$*, and put* $R = S/(z)$*. Let* M *be a finitely generated* R*-module. Given positive integers* p, q*, we have* $\operatorname{redsyz}^p_S(\operatorname{redsyz}^q_R(M)) \cong \operatorname{redsyz}^p_S(\operatorname{syz}^q_R(M)) \cong \operatorname{redsyz}^{p+q}_S(M)$*.*

Proof. Since $\operatorname{syz}^1_S(R) \cong S$, the first isomorphism is clear; therefore we focus on the second. By induction it suffices to treat the case $p = q = 1$. Letting $M_1 = \operatorname{syz}^1_R(M)$ and $m = \mu_R(M)$, we have a short exact sequence of R-modules $0 \to M_1 \xrightarrow{i} R^{(m)} \xrightarrow{\alpha} M \to 0$. We fit this sequence into a commutative exact diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r, "j"] \arrow[d, "\psi"'] & S^{(n)} \arrow[r, "\beta"] \arrow[d, "\phi"'] & M_1 \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & S^{(m)} \arrow[r, "z"] & S^{(m)} \arrow[r, "\pi"] & R^{(m)} \arrow[r] \arrow[d, "\alpha"] & 0 \\
 & & & M \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

Here the top short exact sequence is obtained by mapping some free S-module onto M_1. Thus $\operatorname{redsyz}_S^1(\operatorname{syz}_R^1(M))$ is obtained from N by tossing out all free summands. The map ϕ is a lifting of $i\beta$, and ψ is the induced map on kernels. A routine diagram chase shows that the sequence

$$0 \to N \xrightarrow{\begin{bmatrix} \psi \\ -j \end{bmatrix}} S^{(m)} \oplus S^{(n)} \xrightarrow{[z\ \phi]} S^{(m)} \xrightarrow{\alpha\pi} M \to 0$$

is exact. Thus $\operatorname{redsyz}_S^2(M)$ too is obtained from N by removing free summands. $\square$

Proposition 3.2. (Herzog, [H]) Let R be an indecomposable maximal Cohen-Macaulay module over a Gorenstein local ring $(R, \mathfrak{m}, k)$. Then $\operatorname{syz}_R^n(M)$ is indecomposable for all n.

Proof. For $n = 1$ this is [H, Lemma 1.3]; for $n \geq 2$ we use induction. $\square$

Recall [M, p. 107] that the *multiplicity* of a finitely generated module M over a local ring $(R, \mathfrak{m})$ is defined by $\mathrm{e}(\mathfrak{m}, M) = \lim_{n\to\infty} \frac{d!}{n^d}\ell_R(M/\mathfrak{m}^n M)$, where ℓ_R denotes length. The multiplicity of R is defined by $\mathrm{e}(R) = \mathrm{e}(\mathfrak{m}, R)$. For a hypersurface singularity $R = S/(f)$, where $(S, \mathfrak{n})$ is a regular local ring and $0 \neq f \in \mathfrak{n}$, $\mathrm{e}(R)$ is the largest integer n for which $f \in \mathfrak{n}^n$ (cf. [N, (40.2)]); in particular, $\mathrm{e}(R) = 1$ if and only if R is a regular local ring.

Proposition 3.3. (Kawasaki, [Ka, Theorem 4.1]) *Let $(R, \mathfrak{m})$ be a hypersurface singularity of dimension d and with multiplicity $\mathrm{e}(R) \geq 3$. Then, for every integer $t > \mathrm{e}(R)$, the maximal Cohen-Macaulay R-module $\operatorname{syz}_R^{d+1}(R/\mathfrak{m}^t)$ is indecomposable and requires at least $\binom{d+t-1}{d-1}$ generators.*

Next we review the basic properties of double branched covers. These results could be extracted from Knörrer's paper [Kn], but we will use the exposition in Yoshino's book [Y]. The reader should be aware that Yoshino uses the notation syz^n for the n^{th} *reduced* syzygy. It will be important to us to know that certain syzygies are *automatically* devoid of free direct summands, and thus we need to appeal to Yoshino's proofs rather than merely the statements of his results.

Let $(R, \mathfrak{m}, k)$ be a complete hypersurface singularity, that is, a ring of the form $S/(f)$, where $(S, \mathfrak{n}, k)$ is a complete regular local ring and f is a non-zero element of $\mathfrak{n}$. A *double branched cover* of R is a hypersurface singularity $R^\# := S[[z]]/(f + z^2)$, where z is an indeterminate. Warning: Despite the persuasive notation, $R^\#$ is not always uniquely defined up to isomorphism. For example, $\mathbb{R}[[x,y]]/(x^2) = \mathbb{R}[[x,y]]/(-x^2)$, yet $\mathbb{R}[[x,y,z]]/(z^2 + x^2) \not\cong \mathbb{R}[[x,y,z]](z^2 - x^2)$. Thus, for example, when we write $A \cong R^\#$, we mean that A is isomorphic to the double branched cover of R with respect to *some* presentation $R \cong S/(f)$. This ambiguity is the reason for the occurrence of the units v_i in the definition of (A_1)-singularity.

The element z is a non-zerodivisor on $R^\#$, and by killing z we get a surjective ring homomorphism $R^\# \twoheadrightarrow R$. Thus every R-module can be viewed as an $R^\#$-module. Given a maximal Cohen-Macaulay R-module M, we let $M^\# = \operatorname{syz}^1_{R^\#}(M)$. Since the depth of M is $\dim(R^\#) - 1$, $M^\#$ is a maximal Cohen-Macaulay $R^\#$-module. Also, given a maximal Cohen-Macaulay $R^\#$-module N, we get a maximal Cohen-Macaulay R-module N/zN.

Proposition 3.4. *Let $R = \widehat{R} = S/(f)$ and $R^\# = S[[z]]/(f + z^2)$ as above, let M be a maximal Cohen-Macaulay R-module with no summand isomorphic to R, and let N be a maximal Cohen-Macaulay $R^\#$-module. Then:*

(a) $\operatorname{syz}^1_R(M)$ *has no summand isomorphic to R.*
(b) $M^\#$ *has no summand isomorphic to $R^\#$.*
(c) $M^\#/zM^\# \cong M \oplus \operatorname{syz}^1_R(M)$.
(d) *If* $\operatorname{char}(R) \neq 2$, *then* $(N/zN)^\# \cong N \oplus \operatorname{syz}^1_{R^\#}(N)$.

Proof. For (a), we refer to [Y, Chapter 7]: Since M has no free summand, it is the cokernel of a reduced matrix factorization (φ, ψ). Then $\operatorname{syz}^1_R(M)$ is the cokernel of (ψ, φ) and, by [Y, (7.5.1)], $\operatorname{syz}^1_R(M)$ has no non-zero free summand.

For (c) and (d), we refer to the proofs of (12.4.1) and (12.4.2) in [Y]. The blanket assumption of [Y, Chapter 12], that S is a ring of power series over an algebraically closed field of characteristic 0, is not needed; however the proof of (12.4.2) *does* require that $\frac{1}{2} \in R$.

If (b) were false, we could kill z and get a surjection $M^\#/zM^\# \twoheadrightarrow R$. Since R is local, either M or $\operatorname{syz}^1_R(M)$ would have a non-zero free summand by (c), and this would contradict either the hypotheses or (a). $\square$

The following result from [LW2] (respectively [Kn], [Y, Theorem 12.5]) is an easy consequence:

Corollary 3.5. ([LW2, Proposition 1.5]) *Let $R = \widehat{R} = S/(f)$ and $R^\# = S[[z]]/(f + z^2)$ as above, and assume* $\operatorname{char}(k) \neq 2$. *Then $R^\#$ has bounded (respectively finite) Cohen-Macaulay type if and only if R has bounded (respectively finite) Cohen-Macaulay type.* $\square$

Both here and in §5, we will need the following lemma, whose proof is embedded in the proof of [LW2, Proposition 1.8]:

Lemma 3.6. *Let $(R, \mathfrak{m}, k)$ be a complete hypersurface singularity containing a field, with* $\operatorname{char}(k) \neq 2$. *Assume* $\mathrm{e}(R) = 2$ *and* $d := \dim(R) \geq 2$. *Then there is a complete hypersurface singularity A of dimension $d - 1$ such that $R \cong A^\#$.*

Proof. Write $R = S/(f)$, where $S = k[[x_0, \dots, x_d]]$. Write $f = \sum_{i=0}^{\infty} f_i$, where each f_i is a homogeneous polynomial in $x_0, \dots, x_d$ of degree i. We have $f_0 = f_1 = 0$ and $f_2 \neq 0$. We may assume, after a linear change of variables, that f_2 contains a term of the form cx_d^2, where c is a nonzero element of k. Now consider f as a power series in one variable, x_d, over $S' := k[[x_0, \dots, x_{d-1}]]$. As such, the constant term and the coefficient of x_d are in the maximal ideal of S'. The coefficient of x_d^2 is of the form $c + g$, where g is in the maximal ideal of S'. Therefore, by [Lg, Chapter IV, Theorem 9.2], f can be written uniquely in the form

$$f(x_d) = u(x_d^2 + b_1 x_d + b_2),$$

where the b_i are elements of the maximal ideal of S' and u is a unit of S.

We may ignore the presence of u, as it does not change R. Then, since $\mathrm{char}(k) \neq 2$, we can complete the square and, after a linear change of variables, write $f = x_d^2 + h(x_0, \dots, x_{d-1})$ for some power series $h \in S'$. Putting $A := S'/(h)$, we have $R \cong A^{\#}$. $\square$

Our final task in this section is to prove Theorem 0.1. We will proceed by induction on the dimension, but in order to make the induction proceed more smoothly we will prove a formally strong assertion, which we formulate in Theorem 3.8 below. Let us say that a finitely generated module M over a local ring $(R, \mathfrak{m}, k)$ is *free of constant rank (or constant rank n) on the punctured spectrum* provided there is an integer n such that $M_P \cong R_P^{(n)}$ for every prime ideal $P \neq \mathfrak{m}$. We will need the following "connectedness" result.

Lemma 3.7. *Let $(R, \mathfrak{m}, k)$ be a local ring and T an R-module of finite length. Let $F = \mathrm{redsyz}_R^t(T)$ for some $t \geq 0$. Then F is free of constant rank on the punctured spectrum. If, in addition, R is a complete hypersurface singularity with $\mathrm{e}(R) = 2$ and $\dim(R) \geq 2$, then any direct summand of F is free of constant rank on the punctured spectrum.*

Proof. The first assertion is trivial. For the second, write $R = S/(f)$, where $(S, \mathfrak{n}, k)$ is a regular local ring and $f \in \mathfrak{n}^2 - \mathfrak{n}^3$. Let G be a direct summand of F. Of course G_P is free for every $P \neq \mathfrak{m}$, and the only issue is whether the rank function is constant. If f is irreducible or if $f = ug^2$ for some unit u and some $g \in \mathfrak{n} - \mathfrak{n}^2$, then R has a unique minimal prime ideal Q. Since every (non-maximal) prime P contains Q, we have $\mathrm{rank}(G_P) = \mathrm{rank}(G_Q)$ for all P. The only other possibility is that $f = f_1 f_2$ where f_1 and f_2 are prime elements, neither dividing the other. Now R has two minimal primes $Q_1 = (\overline{f_1})$ and $Q_2 = (\overline{f_2})$. Let $\mathcal{P}$ be any prime ideal of S minimal over (f_1, f_2). Since $\mathcal{P}$ has height 2 and $\dim(S) \geq 3$, $\mathcal{P} \neq \mathfrak{n}$. Then $P := \mathcal{P}/(f)$ is a non-maximal prime ideal of R, and it contains both Q_1 and Q_2. It follows that G has the same rank at Q_1 and at Q_2 and therefore has constant rank on the punctured spectrum. $\square$

Since over a Cohen-Macaulay ring every t^{th} syzygy, for $t \geq \dim(R)$, is maximal Cohen-Macaulay, Theorem 0.1 is an immediate consequence of the following Theorem:

Theorem 3.8. *Let $(R, \mathfrak{m}, k)$ be a hypersurface singularity of dimension $d \geq 1$, containing a field of characteristic different from 2. Suppose $\widehat{R}$ does not have bounded Cohen-Macaulay type. For every integer m, there exist a finite-length R-module T and an integer $t \geq \dim(R)$ such that some direct summand of $\mathrm{redsyz}_R^t(T)$ is indecomposable and is free of constant rank at least m on the punctured spectrum.*

Proof. We may harmlessly assume that $m \geq 2$. Suppose first that R is complete. If $d = 1$, choose any integer $n \geq m$. By [LW3, Proposition 1.1], there is an indecomposable maximal Cohen-Macaulay (= torsion-free) R-module F such that $K \otimes_R F \cong K^{(n)}$, where K is the total quotient ring of R. Thus we get an injection $j : F \to K^{(n)}$ such that j_P is an isomorphism for each non-maximal prime ideal P. Now choose a non-zero-divisor c such that $c \cdot j(F) \subseteq R^{(n)}$. This gives an injection $F \hookrightarrow R^{(n)}$ whose cokernel T has finite length. Since F is indecomposable and $n \geq 2$, we see that $F \cong \operatorname{redsyz}^1_R(T)$ as desired.

Still assuming R is complete, suppose $d \geq 2$. If $\mathrm{e}(R) \geq 3$, we can use Proposition 3.3 to get the required module F. Obviously R is not a regular local ring, so we may assume that $\mathrm{e}(R) = 2$. By Lemma 3.6, $R \cong A^\#$ for a suitable complete hypersurface singularity A of dimension $d-1$. Recall that $A \cong R/(z)$ for some non-zerodivisor z.

By Corollary 3.5, A does not have bounded Cohen-Macaulay type. The inductive hypothesis provides a finite-length A-module T, an integer $t-1 \geq d-1$, and an indecomposable direct summand G of $\operatorname{redsyz}^{t-1}_A(T)$ having constant rank at least $2m$ on the punctured spectrum. Then $G^\# := \operatorname{syz}^1_R(G) = \operatorname{redsyz}^1_R(G)$, by Proposition 3.4. It follows from Lemma 3.1 that $G^\#$ is a direct summand of $\operatorname{redsyz}^t_R(T)$ and therefore, by Lemma 3.7, is free of constant rank on the punctured spectrum. Letting $b = \mu_R(G) = \mu_A(G)$, we have a short exact sequence $0 \to G^\# \to R^{(b)} \to G \to 0$. Localizing at a prime P not containing z, we see that $G^\#_P \cong R^{(b)}_P$. Note that $b \geq 2m$.

By Proposition 3.4, $G^\#/zG^\# \cong G \oplus \operatorname{syz}^1_A(G)$. Since, by Proposition 3.2, $\operatorname{syz}^1_A(G)$ is indecomposable, it follows that $G^\#$ must be a direct sum of at most two indecomposable modules. By Lemma 3.7, $G^\#$ has a direct summand of constant rank at least m on the punctured spectrum. This finishes the proof in the case that R is complete.

In the general case, choose a finite-length $\widehat{R}$-module T, an integer $t \geq \dim(R)$ and an indecomposable direct summand F of $\operatorname{redsyz}^t_{\widehat{R}}(T)$ with constant rank at least m on the punctured spectrum. Then T has finite length as an R-module, and we put $H := \operatorname{syz}^t_R(T)$. Write $H = H_1 \oplus \cdots \oplus H_s$ with each H_i indecomposable. Since $\widehat{H} \cong \operatorname{syz}^t_{\widehat{R}}(T)$, the Krull-Schmidt theorem implies that F is a direct summand of some $\widehat{H_i}$. Moreover, Lemma 3.7 implies that $\widehat{H_i}$ is free of constant rank, say c, on the punctured spectrum of $\widehat{R}$, and of course $c \geq \operatorname{rank}(F) \geq m$. Let p be any non-maximal prime ideal of R, and choose a prime P of $\widehat{R}$ lying over p (cf. [M, Theorem 7.3]). The R_p-module $(H_i)_p$ then becomes free of rank c after the flat local base change $R_p \to \widehat{R}_P$. By faithfully flat descent [EGA, (2.5.8)], $(H_i)_p \cong R^{(c)}_p$. $\square$

4. Finding a suitable finite-length module

The main technical step in the proof of Theorem 0.2 is to find, in dimension one, an indecomposable finite-length module T such that $\operatorname{redsyz}^1(T)$ has large rank. The idea of the construction goes back to the 70's, in papers by Drozd [D] and Ringel [R]. Our development depends on an explicit description, by Klingler and Levy [KL1] of the endomorphism rings of these modules. A *Drozd ring*, [KL1, Definition 2.4], is a commutative Artinian local ring $(\Lambda, \mathcal{M})$ such that $\mu_\Lambda(\mathcal{M}) = \mu_\Lambda(\mathcal{M}^2) = 2$, $\mathcal{M}^3 = 0$, and there is an element $x \in \mathcal{M} - \mathcal{M}^2$ with $x^2 = 0$. The prototype is the ring $k[[x, y]]/(x^2, xy^2, y^3)$ where k is a field.

Lemma 4.1. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with $\dim(R) = 1$ and $\mu_R(\mathfrak{m}) = 2$. If R is not Dedekind-like, then R has a Drozd ring as a homomorphic image.*

Proof. Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. All hypotheses on R transfer to $\widehat{R}$ (cf. [KL3, Lemma 11.8]). Moreover, if we can produce a surjection φ from $\widehat{R}$ onto a Drozd ring Λ, then the composition $R \hookrightarrow \widehat{R} \xrightarrow{\varphi} \Lambda$ is surjective. Therefore we may assume that R is complete.

It will suffice to show that R is not a homomorphic image of a complete local Dedekind-like ring. To see this, we note that R is not a Klein ring (cf. [KL1, Definition 2.8]) since Klein rings are Artinian. Also, since $\mu_R(\mathfrak{m}) = 2$, R does not have an Artinian triad (cf. [KL1, Definition 2.4]) as a homomorphic image. By Klingler and Levy's "dichotomy theorem" [KL1, Theorem 3.1], R maps onto a Drozd ring.

We now assume, by way of contradiction, that D is a complete local Dedekind-like ring and $\sigma : D \twoheadrightarrow R$ is a surjective ring homomorphism.

Suppose first that R is reduced. Of course $\mathrm{Ker}(\sigma) \neq 0$; since both D and R are one-dimensional, D is not a domain. Since the integral closure $\overline{D}$ of D is generated by 2 elements as a D-module, D has exactly two minimal primes P, Q, and both D/P and D/Q are discrete valuation rings. Since R is reduced, either P or Q must be the kernel of σ. But then R is a discrete valuation ring and hence is Dedekind-like, contradiction.

Now assume that R is not reduced. Since $\mathrm{e}(D) \leq 2$ and R and D have the same dimension, it follows that $\mathrm{e}(R) \leq 2$. Since R is Cohen-Macaulay but not a discrete valuation ring, $\mathrm{e}(R)$ must be 2. Write $R = S/I$, where S is a complete regular local ring. Since $\mu_R(\mathfrak{m}) = 2$, we can choose S to be two-dimensional. Since R has depth 1, the Auslander-Buchsbaum formula [M, Theorem 19.1] says that R has projective dimension one as an S-module. Thus I is principal, say, $I = Sf$, where $f \in \mathfrak{n}^2 - \mathfrak{n}^3$. Since R is not reduced, we have, up to a unit, $f = x^2$, where $x \in \mathfrak{n} - \mathfrak{n}^2$. Choosing an element $y \in S$ such that $\mathfrak{n} = (x, y)$, we see that R maps onto the Drozd ring $S/(x^2, xy^2, y^3)$. $\square$

Lemma 4.2. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay local ring with $\mu_R(\mathfrak{m}) = 2$. Assume R is not Dedekind-like. Given any integer n, there is an indecomposable finite-length module T such that $F := \mathrm{redsyz}^1_R(T)$ is free of constant rank greater than n on the punctured spectrum.*

Proof. Choose, using Lemma 4.1, an ideal I such that $\Lambda := R/I$ is a Drozd ring. Fix elements $x, y \in \mathfrak{m}$ such that $\mathfrak{m} = Rx + Ry$ and $x^2 \in I$. When there is no danger of confusion we denote the images of these elements in Λ simply by x and y.

Fix a positive integer n, and let ϕ be the $n \times n$ invertible matrix (over R or Λ) with 1's on the diagonal and superdiagonal and 0's elsewhere. We will follow the development in [KL1] closely, with the exception that our matrices act on the left and we write vectors in $\Lambda^{(n)}$ as columns. Put

$$Q := \frac{\Lambda^{(n)}}{y\Lambda^{(n)}} \oplus \frac{\Lambda^{(n)}}{xy\Lambda^{(n)}} \oplus \Lambda^{(n)}, \tag{4.2.1}$$

and let $\mathcal{R}$ denote the R-submodule of Q consisting of elements of the form

$$(\mathbf{b}x + y\Lambda^{(n)}, -\mathbf{b}y^2 - \mathbf{d}y + \mathbf{c}x + xy\Lambda^{(n)}, \mathbf{d}x - (\phi\mathbf{c})y^2), \tag{4.2.2}$$

where $\mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ range over $\Lambda^{(n)}$. Finally, put $T := Q/\mathcal{R}$. Of course T is a torsion R-module, since it is killed by $\mathfrak{m}^3$.

To show that T is indecomposable, suppose f is an idempotent endomorphism of T. We will show that f is either 0 or 1. Let $\Gamma = \{g \in \operatorname{End}_\Lambda(\Lambda^{(n)}) \mid g(\mathcal{R}) \subseteq \mathcal{R}\}$. Since the obvious surjection $\sigma : \Lambda^{(3n)} \to T$ is a projective cover, the induced map $\Gamma \to \operatorname{End}_\Lambda(T)$ is surjective, and by Lemma 2.4 its kernel is contained in $\mathrm{J}(\Gamma)$. Since Γ is left Artinian, idempotents lift modulo the Jacobson radical (cf. [Lm, (4.12), (21.28)]). Thus let $F \in \Gamma$ be an idempotent lifting f. It will suffice to show that F is either 0 or 1. Now we invoke [KL1, Lemma 4.8], which implies that F has the following block form:

$$F = \begin{bmatrix} F_{11} & * & * \\ \alpha & F_{22} & * \\ \beta & \gamma & F_{33} \end{bmatrix},$$

where

(1) each block is an $n \times n$ matrix,
(2) $F_{11} \equiv F_{22} \equiv F_{33} \pmod{\mathcal{M}}$,
(3) $\phi F_{11} \equiv F_{11}\phi \pmod{\mathcal{M}}$, and
(4) the entries of α, β and γ are in $\mathcal{M}$.

(Our matrix is the transpose of the matrix displayed in [KL1, (4.8.1)], since ours operates on the left.)

Lettings bars denote reduction modulo $\mathcal{M}$, we have

$$\overline{F} = \begin{bmatrix} \overline{F_{11}} & * & * \\ 0 & \overline{F_{11}} & * \\ 0 & 0 & \overline{F_{11}} \end{bmatrix}.$$

Since $\overline{F_{11}}$ commutes with the non-derogatory matrix $\overline{\phi}$, $\overline{F_{11}}$ belongs to $k[\overline{\phi}]$, which is a local ring. Moreover, since $\overline{F}^2 = \overline{F}$, it follows that $\overline{F_{11}}^2 = \overline{F_{11}}$. Therefore $\overline{F_{11}}^2 = 0$ or 1. An easy computation then shows that $\overline{F} = 0$ or 1. By Lemma 2.4 the kernel of the map $\operatorname{End}_\Lambda(\Lambda^{(3n)}) \to \operatorname{End}_k(k^{(3n)})$ is contained in the Jacobson radical of $\operatorname{End}_\Lambda(\Lambda^{(3n)})$. It follows that $F = 0$ or 1, as desired.

Let $L := \operatorname{syz}^1_R(T)$, and write $L = R^{(r)} \oplus F$, where F has no non-zero free direct summand. To complete the proof, it will suffice to show that $\operatorname{rank}(F) \geq \frac{n}{e-1}$, where $e = \mathrm{e}_R(R)$. Put $s := \operatorname{rank}(F)$ and $m := \mu_R(F)$. It follows, e.g., from [HW, (1.6)], that $m \leq es$. (The statement of [HW, (1.6)] assumes that k is infinite. This is not a problem, since none of m, e, s is changed by the flat local base change $R \to R(X) := R[X]_{\mathfrak{m}[X]}$.) Now $\mu_R(L) = r + m = 3n - s + m$, whence $\mu_R(L) - 3n \leq (e-1)s$. Therefore it will suffice to show that $\mu_R(L) \geq 4n$. Since $\mu_R(\mathfrak{m}) = 2$, the following lemma completes the proof:

Lemma 4.3. *There is a surjective R-homomorphism from L onto $\mathfrak{m}^{(2n)}$.*

Proof. Let Q be as in (4.2.1), and let $\rho : R^{(n)} \oplus R^{(n)} \oplus R^{(n)} \to Q$ be the natural homomorphism. Then $L = \rho^{-1}(\mathcal{R})$. Let $\pi : R^{(n)} \oplus R^{(n)} \oplus R^{(n)} \to R^{(n)} \oplus R^{(n)}$ be the projection on the first two coordinates. We will show that $\pi(L) = \mathfrak{m}^{(n)} \oplus \mathfrak{m}^{(n)}$. Since $\Lambda^{(3n)} \to T$ is a projective cover [KL1, (4.6.4)], $\mu_R(T) = 3n$. Therefore $L \subseteq \mathfrak{m}(R^{(n)} \oplus R^{(n)} \oplus R^{(n)})$, and it follows that $\pi(L) \subseteq \mathfrak{m}^{(n)} \oplus \mathfrak{m}^{(n)}$.

For the reverse inclusion, fix $i, 1 \le i \le n$, and let $\mathbf{e}_i \in R^{(n)}$ be the i^{th} unit vector. It will suffice to show that the four elements $(\mathbf{e}_i x, 0), (\mathbf{e}_i y, 0), (0, \mathbf{e}_i x)$ and $(0, \mathbf{e}_i y)$ are all in $\pi(L)$.

We have $(\mathbf{e}_i x, 0) = \pi(\mathbf{e}_i x, 0, -\mathbf{e}_i x)$, and clearly $(\mathbf{e}_i x, 0, -\mathbf{e}_i x) \in L$. (Take the elements $\mathbf{b}, \mathbf{c}, \mathbf{d}$ in (4.2.2) to be the images, in $\Lambda^{(n)}$, of $\mathbf{e}_i, 0, -\mathbf{e}_i y$, respectively.) Since $\rho(y\mathbf{e}_i, 0, 0) = 0 \in \mathcal{R}$, $(\mathbf{e}_i y, 0) \in \pi(L)$. Next, we have $(0, \mathbf{e}_i x) = \pi(0, \mathbf{e}_i x, -(\phi \mathbf{e}_i) y^2) \in \pi(L)$. (Take $\mathbf{c}$ to be the image of $\mathbf{e}_i$, and take $\mathbf{b} = \mathbf{d} = 0$.) Finally, $(0, \mathbf{e}_i y) = \pi(0, \mathbf{e}_i y, -\mathbf{e}_i x) \in \pi(L)$. (Take $\mathbf{b} = \mathbf{c} = 0$, and let $\mathbf{d}$ be the image of $-\mathbf{e}_i$.) This completes the proof of Lemma 4.3, and therefore of Lemma 4.2 as well. $\square$

5. The main application

We begin with three preparatory lemmas, the first of which is an iterated version of Lemma 3.6.

Lemma 5.1. *Let $(R, \mathfrak{m}, k)$ be a complete hypersurface singularity containing a field of characteristic different from 2. Assume $d := \dim(R) \ge 2$ and that R has bounded Cohen-Macaulay type. Then R is isomorphic to a ring of the form*

$$(\dagger) \qquad k[[x_0, \dots, x_d]]/(g + v_1 x_2^2 + v_1 v_2 x_3^2 + \dots + v_1 v_2 \cdot \dots \cdot v_{d-1} x_d^2),$$

where each v_i is a unit of $k[[x_0, \dots, x_i]]$ and $g \in k[[x_0, x_1]]$. Moreover, if we put $R_1 := k[[x_0, x_1]]/(g)$ and $R_i := k[[x_0, x_1, \dots, x_i]]/(g + v_1 x_2^2 + v_1 v_2 x_3^2 + \dots + v_1 v_2 \cdot \dots \cdot v_{i-1} x_i^2)$ for $2 \le i \le d$, we have $R_i \cong R_{i-1}^{\#}$ for $2 \le i \le d$.

Proof. By Proposition 3.3, $\mathrm{e}(R) \le 2$. Therefore $\mathrm{e}(R) = 2$ since R is not a regular local ring. Write $R_d = k[[x_0, \dots, x_d]]/(f)$. As in the proof of Lemma 3.6, we can do a linear change of variables to get $f = u_d(x_d^2 + g_{d-1})$, where u_d is a unit and $g_{d-1} \in k[[x_0, \dots, x_{d-1}]]$. With $A = k[[x_0, \dots, x_{d-1}]]/(g_{d-1})$, we see that $R_d \cong A^{\#}$. By Corollary 3.5, A has bounded Cohen-Macaulay type. Also, $g_{d-1} \in (x_0, \dots, x_{d-1})^2$ (else R would be regular), so A is not regular. Continuing (if $d \ge 3$), we note that the next change of variables, in $k[[x_0, \dots, x_{d-1}]]$, does not affect x_d. Eventually, we get units $u_i \in k[[x_0, \dots, x_i]]$ and $g_1 \in k[[x_0, x_1]]$ such that

$$R \cong k[[x_0, \dots, x_d]]/u_d(x_d^2 + u_{d-1}(x_{d-1}^2 + u_{d-2}(\dots(x_3^2 + u_2(x_2^2 + g_1))\dots))).$$

Let $v_i = u_i^{-1}$ for each i, and put $v_1 = 1$. Multiplying the defining equation by $v_1 v_2 \cdot \dots \cdot v_d$ and putting $g = g_1$, we obtain the desired form. The "Moreover" assertion is clear, once we multiply the defining equation for R_i by $(v_1 \cdot \dots \cdot v_{i-1})^{-1}$. $\square$

Lemma 5.2. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring, M a finitely generated R-module and F a maximal Cohen-Macaulay R-module. Put $B = \mathrm{End}_R(F)$. Then, for all integers $i \ge 0$ and $j \ge 1$, we have*

$$(5.2.1) \quad \mathrm{Ext}_R^{i+j}(F, \mathrm{redsyz}_R^i(M)) \cong \mathrm{Ext}_R^{i+j}(F, \mathrm{syz}_R^i(M)) \cong \mathrm{Ext}_R^j(F, M) \text{ as right } B\text{-modules.}$$

Proof. Since R is Gorenstein and F is maximal Cohen-Macaulay, we have $\operatorname{Ext}^j_R(F,R)=0$ for $j \geq 1$. Thus we may as well use actual syzygies instead of reduced syzygies. The desired isomorphism is obtained inductively, by applying $\operatorname{Hom}_R(F, \_\_)$ to the short exact sequences $0 \to \operatorname{syz}^{j+1}_R(M) \to R^{(n_j)} \to \operatorname{syz}^j_R(M) \to 0$. The resulting isomorphisms are B-linear, by naturality of the connecting homomorphisms in the long exact sequence of Ext. $\square$

Lemma 5.3. ([Y, (7.2)]) *Let $(R, \mathfrak{m}, k)$ be a complete hypersurface singularity, and let M be a maximal Cohen-Macaulay R-module having no non-zero free summand. Then M has a periodic minimal free resolution, with period at most* 2. $\square$

Finally, we state and prove Theorem 0.2 in the following slightly stronger form:

Theorem 5.4. *Let $(R, \mathfrak{m}, k)$ be a hypersurface singularity of dimension $d \geq 1$, containing a field of characteristic different from 2. Assume that $\widehat{R}$ has bounded Cohen-Macaulay type but is not an (A_1)-singularity. Put $t = d$ if d is odd and $t = d+1$ if d is even. Given any positive integer m there is a short exact sequence of finitely generated R-modules*

$$0 \to T \to X \to F \to 0, \tag{5.4.1}$$

in which

(a) *T is an indecomposable finite-length module,*
(b) *X is indecomposable,*
(c) *$F \cong \operatorname{redsyz}^t_R(T)$, and*
(d) *F and X are free of (the same) constant rank at least m on the punctured spectrum.*

Proof. We may assume that $m \geq 2$. Suppose for the moment that we have proved the theorem in the complete case, and let T, X, F be $\widehat{R}$-modules fitting into the exact sequence (5.4.1) and satisfying (a) – (d) (for $\widehat{R}$). Write $F \oplus \widehat{R}^{(b)} \cong \operatorname{syz}^t_{\widehat{R}}(T)$. In the general case, let $H = \operatorname{redsyz}^t_R(T)$, and write $H \oplus R^{(a)} \cong \operatorname{syz}^t_R(T)$. Then $\widehat{H} \oplus \widehat{R}^{(b)} \cong \operatorname{syz}^t_{\widehat{R}}(T)$. Since R is not isomorphic to a direct summand of H, it follows, e.g., from [W, Proposition 2], that $\widehat{R}$ is not isomorphic to a direct summand of $\widehat{H}$. Therefore $\widehat{H} \cong \operatorname{redsyz}^t_{\widehat{R}}(T) \cong F$. Since $\operatorname{Ext}^1_R(T,H)$ has finite length as an R-module, we have $\operatorname{Ext}^1_R(T,H) = (\operatorname{Ext}^1_R(T,H))\widehat{\ } = \operatorname{Ext}^1_{\widehat{R}}(\widehat{T}, \widehat{H}) = \operatorname{Ext}^1_{\widehat{R}}(T,F)$. This means that the extension (5.4.1) over $\widehat{R}$ is actually the completion of an extension $0 \to T \to Y \to H \to 0$ of R-modules. It follows that $\widehat{Y} \cong X$ and hence that Y is indecomposable. Finally, the argument in the last three sentences of §3 shows that Y and H are free on the punctured spectrum, of the same rank as X and F.

Thus we may assume from now on that R is complete. We write $R = R_d$ in the form (†), using Lemma 5.1. With the R_i as in Lemma 5.1, we make the identifications $R_i = R^{\#}_{i-1}$ and $R_{i-1} = R_i/(z_i)$. None of the rings R_i is an (A_1)-singularity; in particular, R_1 is not Dedekind-like. By Lemma 4.2, there is a finite-length R_1-module T whose first reduced syzygy F_1 is free of constant rank at least m on the punctured spectrum.

We now define R_i-modules F_i inductively, for $i = 2, \ldots, d$, by letting $F_i = F^{\#}_{i-1}$ ($= \operatorname{syz}^1_{R_i}(F_{i-1})$). Applying Proposition 3.4 and Lemma 3.1 inductively, we see that F_i has no non-zero free direct summand and that

$$F_i \cong \operatorname{redsyz}^i_{R_i}(T) \text{ for } i = 1, \ldots, d. \tag{5.4.2}$$

Therefore F_i is free of constant rank on the punctured spectrum of R_i. To estimate the size of this rank, we look at the short exact sequence $0 \to F_i \to R^{(b_{i-1})} \to F_{i-1} \to 0$, where $b_{i-1} = \mu_{R_{i-1}}(F_{i-1})$. By localizing at a prime ideal P not containing z_i, we learn that the rank of F_i is exactly b_{i-1}. Since a module that is free of rank r on the punctured spectrum obviously needs at least r generators, we have inequalities $b_{d-1} \geq \cdots \geq b_1 \geq m$.

Next, we let $G_1 = \operatorname{syz}^1_{R_1}(F_1)$. By Proposition 3.4 G_1 has no non-zero free direct summand, and $\mu_{R_1}(G_1) = \mu_{R_1}(F_1) = b_1$ by Lemma 5.3. For $i = 2, \ldots, d$ we define $G_i = G^{\#}_{i-1}$ ($= \operatorname{syz}^1_{R_i}(G_{i-1})$). By Proposition 3.4, G_i has no non-zero free summands, that is, $G_i = \operatorname{redsyz}^1_{R_i}(G_{i-1})$. Using Lemma 3.1, we see that

$$G_i = \operatorname{redsyz}^1_{R_i}(F_i), \text{ for } i = 1, \ldots, d. \tag{5.4.3}$$

The argument in the last paragraph shows that the rank of G_i (on the punctured spectrum of R_i) is at least m, for $i = 2, \ldots, d$. (Fortunately, we don't care about the rank of G_1.)

Recall that $R = R_d$. Suppose first that d is odd (possibly $d = 1$). We put $F := F_d$ and $B := \operatorname{End}_R(F)$. Since d is odd, we have, by periodicity (Lemma 5.3),

$$F \cong \operatorname{syz}^d_R(G_d) \cong \operatorname{redsyz}^d_R(G_d). \tag{5.4.4}$$

Applying Lemma 5.2 to (5.4.2) and (5.4.4), we obtain isomorphisms of right B-modules

$$\operatorname{Ext}^1_R(F, T) \cong \operatorname{Ext}^{d+1}_R(F, F) \cong \operatorname{Ext}^1_R(F, G_d). \tag{5.4.5}$$

By (5.4.3), there is a short exact sequence

$$0 \to G_d \to R^{(b)} \xrightarrow{\varphi} F \to 0,$$

where $b = b_d = \mu_R(F)$. Applying $\operatorname{Hom}_R(F, \_\_)$, we get an exact sequence of B-modules

$$\operatorname{Hom}_R(F, R^{(b)}) \xrightarrow{\varphi_*} B \xrightarrow{\delta} \operatorname{Ext}^1_R(F, G_d).$$

Combining this with (5.4.5), we obtain an exact sequence of right B-modules

$$\operatorname{Hom}_R(F, R^{(b)}) \xrightarrow{\varphi_*} B \xrightarrow{\beta} \operatorname{Ext}^1_R(F, T).$$

If $f : F \to F$ is in the image of φ_*, then $f(F) \subseteq \mathfrak{m}F$, as F has no non-zero free summands. By Lemma 2.4, $\operatorname{Ker}(\beta) \subseteq \mathrm{J}(B)$, and now Theorem 2.3 provides the desired exact sequence (5.4.1).

If d is even, then $\operatorname{syz}^{d+1}_R(F_d) \cong G_d$ by periodicity (Lemma 5.3). But $G_d \cong \operatorname{redsyz}^{d+1}_{R_d}(T)$ by Lemma 3.1. Two applications of Lemma 5.2 now show that $\operatorname{Ext}^1_R(G_d, F_d) \cong \operatorname{Ext}^1_R(G_d, T)$

as right $\mathrm{End}_R(G_d)$-modules. Therefore, when we apply $\mathrm{Hom}_R(G_d, \_\_)$ to the short exact sequence

$$0 \to F_d \to R^{(t)} \xrightarrow{\psi} G_d \to 0,$$

we obtain an exact sequence

$$\mathrm{Hom}_R(G_d, R^{(t)}) \xrightarrow{\psi_*} \mathrm{End}_R(G_d) \xrightarrow{\beta} \mathrm{Ext}^1_R(G_d, T)$$

of right $\mathrm{End}_R(G_d)$-modules. We put $F = G_d$ and proceed exactly as in the case where d is odd. $\square$

We conclude with the following result, a reformulation of the main results of [LW2]:

Corollary 5.5. *Let $(R, \mathfrak{m}, k)$ be a complete hypersurface singularity containing a field of characteristic different from 2. Then R has bounded but infinite Cohen-Macaulay type if and only if R is isomorphic to a ring of the form (†), where g is either x_1^2 or $x_0x_1^2$.*

Proof. By [BGS, Proposition 4.2] (cf. also [LW2]), $k[[x_0, x_1]]/(x_1^2)$ and $k[[x_0, x_1]]/(x_0x_1^2)$ have bounded but infinite Cohen-Macaulay type. The "if" direction now follows from Lemma 5.1 and Corollary 3.5.

For the converse, suppose R has bounded but infinite Cohen-Macaulay type. Using Lemma 5.1, we can put R into the form (†). By Corollary 3.5, the ring $A := k[[x_0, x_1]]/(g)$ has bounded but infinite Cohen-Macaulay type. The arguments in [LW2] show that after a change of variables in $k[[x_0, x_1]]$ we have either $g = ux_1^2$ or $g = ux_0x_1^2$ for some unit $u \in k[[x_0, x_1]]$. Now multiply the defining equation for R by u^{-1}, and replace v_1 by $u^{-1}v_1$, to get the desired form. $\square$

References

[BGS] R.-O. Buchweitz, G.-M. Greuel and F.-O. Schreyer, *Cohen-Macaulay modules on hypersurface singularities II*, Invent. Math. **88** (1987), 165–182.

[D] Yu. A. Drozd, *Representations of commutative algebras (Russian)*, Funktsional. Anal. i Prilozhen. **6** (1972), 41–43; English Translation in Funct. Anal. Appl. **6** (1972), 286–288.

[E] E. G. Evans, Jr., *Krull-Schmidt and cancellation over local rings*, Pacific J. Math. **46** (1973), 115–121.

[EGA] A. Grothendieck and J. Dieudonné, *Éléments de Géométrie Algébrique IV, Partie 2*, Publ. Math. I.H.E.S., 1967.

[F] A. Facchini, *Module Theory*, Birkhäuser Verlag, Basel, 1998.

[HKKW] W. Hassler, R. Karr, L. Klingler and R. Wiegand, *Indecomposable modules of large rank over commutative local rings*, preprint.

[H] J. Herzog, *Ringe mit nur endlich vielen Isomorphieklassen von maximalen unzerlegbaren Cohen–Macaulay Moduln*, Math. Ann. **233** (1978), 21–34.

[HW] C. Huneke and R. Wiegand, *Tensor products of modules and the rigidity of* Tor, Math. Ann. **299** (1994), 449–476.

[Ka] T. Kawasaki, *Local cohomology modules of indecomposable surjective-Buchsbaum modules over Gorenstein local rings.*, J. Math. Soc. Japan **48**, 551–566.

[Kn] H. Knörrer, *Cohen-Macaulay modules on hypersurface singularities I*, Invent. Math. **88** (1987), 153–164.

[KL1] L. Klingler and L. S. Levy, *Representation type of commutative Noetherian rings I: Local wildness*, Pacific J. Math. **200** (2001), 345–386.

[KL2] L. Klingler and L. S. Levy, *Representation type of commutative Noetherian rings II: Local tameness*, Pacific J. Math. **200** (2001), 387–483.

[KL3] L. Klingler and L. S. Levy, *Representation type of commutative Noetherian rings III: Global wildness and tameness*, Mem. Amer. Math. Soc. (to appear).

[Lm] T. Y. Lam, *A First Course in Noncommutative Rings*, Second Edition, Graduate Texts in Math., vol. 131, Springer, New York, 2001.

[Lg] S. Lang, *Algebra, 3rd. ed.*, Addison-Wesley, 1993.

[LW1] G. Leuschke and R. Wiegand, *Ascent of finite Cohen-Macaulay type*, J. Algebra **228** (2000), 674–681.

[LW2] ———, *Hypersurfaces of bounded Cohen-Macaulay type*, J. Pure Appl. Algebra (to appear).

[LW3] ———, *Local rings of bounded Cohen-Macaulay type*, Algebr. Represent. Theory (to appear).

[M] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, Cambridge, 1989.

[N] M. Nagata, *Local Rings*, Interscience, New York, 1962.

[R] K. M. Ringel, *The representation type of local algebras*, Lecture Notes in Math. **488** (1975), Springer, New York, 282–305.

[W] R. Wiegand, *Direct-sum decompositions over local rings*, J. Algebra **240** (2001), 83–97.

[Y] Y. Yoshino, *Cohen-Macaulay Modules over Cohen-Macaulay Rings*, London Math. Soc. Lect. Notes **146**, 1990.