A CHANGE OF RINGS RESULT FOR MATLIS REFLEXIVITY

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Abstract. Let R be a commutative Noetherian ring and E the minimal injective cogenerator of the category of R-modules. An R-module M is (Matlis) reflexive if the natural evaluation map $M \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M, E), E)$ is an isomorphism. We prove that if S is a multiplicatively closed subset of R and M is an R_S-module which is reflexive as an R-module, then M is a reflexive R_S-module. The converse holds when S is the complement of the union of finitely many nonminimal primes of R, but fails in general.

1. Introduction

Let R be a commutative Noetherian ring and E the minimal injective cogenerator of the category of R-modules; i.e., $E = \bigoplus_{m\in\Lambda} E_R(R/m)$, where Λ denotes the set of maximal ideals of R and $E_R(-)$ denotes the injective hull. An R-module M is said to be *(Matlis) reflexive* if the natural evaluation map $M \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M, E), E)$ is an isomorphism. In [1], the authors assert the following "change of rings" principal for Matlis reflexivity ([1, Lemma 2]): *Let S be a multiplicatively closed subset of R and suppose M is an R_S-module. Then M is reflexive as an R-module if and only if M is reflexive as an R_S-module.* However, the proof given in [1] is incorrect (see Examples 3.1-3.3) and in fact the "if" part is false in general (cf. Proposition 3.4). In this note, we prove the following:

Theorem 1.1. *Let R be a Noetherian ring, S a multiplicatively closed subset of R, and M an R_S-module.*

(a) If M is reflexive as an R-module then M is reflexive as an R_S-module.
(b) If $S = R \setminus (p_1 \cup \ldots \cup p_r)$ where each p_i is a maximal ideal or a nonminimal prime ideal, then the converse to (a) holds.

2. Main results

Throughout this section R will denote a Noetherian ring and S a multiplicatively closed subset of R. We let E_R (or just E if the ring is clear) denote the minimal injective cogenerator of the category of R-modules as defined in the introduction. A semilocal ring is said to be complete if it is complete with respect to the J-adic topology, where J is the Jacobson radical.

We will make use of the main result of [1]:

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Theorem 2.1. ([1, Theorem 12]) *Let R be a Noetherian ring, M an R-module, and $I = \operatorname{Ann}_R M$. Then M is reflexive if and only if R/I is a complete semilocal ring and there exists a finitely generated submodule N of M such that M/N is Artinian.*

We remark that the validity of this theorem does not depend on [1, Lemma 2], as the proof of [1, Theorem 12] uses this lemma only in a special case where it is easily seen to hold. (See the proof of [1, Theorem 9], which is the only instance [1, Lemma 2] is used critically.)

Lemma 2.2. ([1, Lemma 1]) *Let M be an R-module and I an ideal of R such that $IM = 0$. Then M is reflexive as an R-module if and only if M is reflexive as an R/I-module.*

Proof. Since $E_{R/I} = \operatorname{Hom}_R(R/I, E_R)$, the result follows readily by Hom-tensor adjunction. $\square$

Lemma 2.3. *Let $R = R_1 \times \cdots \times R_k$ be a product of Noetherian local rings. Let $M = M_1 \times \cdots \times M_k$ be an R-module. Then M is reflexive as an R-module if and only if M_i is reflexive as an R_i-module for all i.*

Proof. As finite sums and direct summands of reflexive modules are reflexive, it suffices to prove that M_i is reflexive as an R-module if and only if M_i is reflexive as an R_i-module for each i. But this follows immediately from Lemma 2.2. $\square$

Theorem 2.4. *Let S be a multiplicatively closed subset of R and M an R_S-module which is reflexive as an R-module. Then M is reflexive as an R_S-module.*

Proof. By Lemma 2.2, we may assume $\operatorname{Ann}_R M = \operatorname{Ann}_{R_S} M = 0$. Thus, R is semilocal and complete by Theorem 2.1. Hence, $R = R_1 \times \cdots \times R_k$ where each R_i is a complete local ring. Then $R_S = (R_1)_{S_1} \times \cdots \times (R_k)_{S_k}$ where S_i is the image of S under the canonical projection $R \longrightarrow R_i$. Write $M = M_1 \times \cdots \times M_k$, where $M_i = R_i M$. As M is reflexive as an R-module, M_i is reflexive as an R_i-module for all i. Thus, it suffices to show that M_i is reflexive as an $(R_i)_{S_i}$-module for all i. Hence, we may reduce the proof to the case (R, m) is a complete local ring with $\operatorname{Ann}_R M = 0$ by passing to $R/\operatorname{Ann}_R M$, if necessary. As M is reflexive as an R-module, we have by Theorem 2.1 that there exists an exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow X \longrightarrow 0$$

where N is a finitely generated R-module and X is an Artinian R-module. If $S \cap m = \emptyset$, then $R_S = R$ and there is nothing to prove. Otherwise, as $\operatorname{Supp}_R X \subseteq \{m\}$, we have $X_S = 0$. Hence, $M \cong N_S$, a finitely generated R_S-module. To see that M is R_S-reflexive, it suffices to show that R_S is Artinian (hence semilocal and complete). Since $\operatorname{Ann}_R N_S = \operatorname{Ann}_R M = 0$, we have that $\operatorname{Ann}_R N = 0$. Thus, $\dim R = \dim N$. Since M is an R_S-module and $S \cap m \neq \emptyset$, we have $H^i_m(M) \cong H^i_{mR_S}(M) = 0$ for all i. Further, as X is Artinian, $H^i_m(X) = 0$ for $i \geq 1$. Thus, from the long exact sequence on local cohomology, we conclude that $H^i_m(N) = 0$ for $i \geq 2$. Thus, $\dim R = \dim N \leq 1$, and hence, $\dim R_S = 0$. Consequently, R_S is Artinian, and M is a reflexive R_S-module. $\square$

To prove part (b) of Theorem 1.1, we will need the following result on Henselian local rings found in [2] (in which the authors credit it to F. Schmidt). As we need a slightly different version of this result than what is stated in [2] and the proof is short, we include it for the convenience of the reader:

Proposition 2.5. ([2, Satz 2.3.11]) *Let (R, m) be a local Henselian domain which is not a field and F the field of fractions of R. Let V be a discrete valuation ring with field of fractions F. Then $R \subseteq V$.*

Proof. Let k be the residue field of R and $a \in m$. As R is Henselian, for every positive integer n not divisible by the characteristic of k, the polynomial $x^n - (1+a)$ has a root b in R. Let v be the valuation on F associated to V. Then $nv(b) = v(1+a)$. If $v(a) < 0$ then $v(1+a) < 0$ which implies $v(b) \leq -1$. Hence, $v(1+a) \leq -n$. As n can be arbitrarily large, this leads to a contradiction. Hence, $v(a) \geq 0$ and $a \in V$. Thus, $m \subseteq V$. Now let $c \in R$ be arbitrary. Choose $d \in m, d \neq 0$. If $v(c) < 0$ then $v(c^\ell d) < 0$ for ℓ sufficiently large. But this contradicts that $c^\ell d \in m \subseteq V$ for every ℓ. Hence $v(c) \geq 0$ and $R \subseteq V$. □

For a Noetherian ring R, let $\operatorname{Min} R$ and $\operatorname{Max} R$ denote the set of minimal and maximal primes of R, respectively. Let $\mathrm{T}(R) = (\operatorname{Spec} R \setminus \operatorname{Min} R) \cup \operatorname{Max} R$.

Lemma 2.6. *Let R be a Noetherian ring and $p \in \mathrm{T}(R)$. If R_p is Henselian then the natural map $\varphi : R \longrightarrow R_p$ is surjective; i.e., $R/\ker\varphi \cong R_p$.*

Proof. By replacing R with $R/\ker\varphi$, we may assume φ is injective. Then p contains every minimal prime of R. Let $u \in R, u \notin p$. It suffices to prove that the image of u in R/q is a unit for every minimal prime q of R. Hence, we may assume that R is a domain. (Note that $(R/q)_p = R_p/qR_p$ is still Henselian.) If R_p is a field, then, as $p \in \mathrm{T}(R)$, we must have R is a field (as p must be both minimal and maximal in a domain). So certainly $u \notin p = (0)$ is a unit in R. Thus, we may assume R_p is not a field. Suppose u is not a unit in R. Then $u \in n$ for some maximal ideal n of R. Now, there exists a discrete valuation ring V with same field of fractions as R such that $m_V \cap R = n$ ([5, Theorem 6.3.3]). As R_p is Henselian, $R_p \subseteq V$ by Proposition 2.5. But as $u \notin p$, u is a unit in R_p, hence in V, contradicting $u \in n \subseteq m_V$. Thus, u is a unit in R and $R = R_p$. □

Proposition 2.7. *Let R be a Noetherian ring and $S = R \setminus (p_1 \cup \cdots \cup p_r)$ where $p_1, \ldots, p_r \in \mathrm{T}(R)$. Suppose R_S is complete with respect to its Jacobson radical. Then the natural map $\varphi : R \longrightarrow R_S$ is surjective.*

Proof. First, we may assume that $p_j \nsubseteq \bigcup_{i \neq j} p_i$ for all j. Also, by passing to the ring $R/\ker\varphi$, we may assume φ is injective. (We note that if $p_{i_1}, \ldots, p_{i_t}$ are the ideals in the set $\{p_1, \ldots, p_r\}$ containing $\ker\varphi$, it is easily seen that $(R/\ker\varphi)_S = (R/\ker\varphi)_T$ where $T = R \setminus (p_{i_1} \cup \cdots \cup p_{i_t})$. Hence, we may assume each p_i contains $\ker\varphi$.) As R_S is semilocal and complete, the map $\psi : R_S \longrightarrow R_{p_1} \times \cdots \times R_{p_r}$ given by $\psi(u) = (\frac{u}{1}, \ldots, \frac{u}{1})$ is an isomorphism. For each i, let $\rho_i : R \longrightarrow R_{p_i}$ be the natural map. Since $R \longrightarrow R_S$ is an injection, $\cap_i \ker\rho_i = (0)$. It suffices to prove that u is a unit in R for every $u \in S$. As R_{p_i} is complete, hence Henselian, we have that ρ_i is surjective for each i by Lemma 2.6. Thus, u is a unit in $R/\ker\rho_i$ for every i; i.e., $(u) + \ker\rho_i = R$ for $i = 1, \ldots, r$. Then $(u) = (u) + (\cap_i \ker\rho_i) = R$. Hence, u is a unit in R. □

We now prove part (b) of the Theorem 1.1:

Theorem 2.8. *Let R be a Noetherian ring and M a reflexive R_S-module, where S is the complement in R of the union of finitely many elements of $\mathrm{T}(R)$. Then M is reflexive as an R-module.*

Proof. We may assume $M \neq 0$. Let $S = R \setminus (p_1 \cup \cdots \cup p_r)$, where $p_1, \ldots, p_r \in \mathrm{T}(R)$ Let $I = \operatorname{Ann}_R M$, whence $I_S = \operatorname{Ann}_{R_S} M$. As in the proof of Proposition 2.7, we may assume each p_i contains I. Then by Lemma 2.2, we may reduce to the case $\operatorname{Ann}_R M = \operatorname{Ann}_{R_S} M = 0$. Note that this implies the natural map $R \longrightarrow R_S$ is injective. As M is R_S-reflexive, R_S is complete with respect to its Jacobson radical by Theorem 2.1. By Proposition 2.7, we have that $R \cong R_S$ and hence M is R-reflexive. $\square$

3. Examples

The following examples show that $\operatorname{Hom}_R(R_S, E_R)$ need not be the minimal injective cogenerator for the category of R_S-modules, contrary to what is stated in the proof of [1, Lemma 2]:

Example 3.1. Let (R, m) be a local ring of dimension at least two and p any prime which is not maximal or minimal. By [3, Lemma 4.1], every element of $\operatorname{Spec} R_p$ is an associated prime of the R_p-module $\operatorname{Hom}_R(R_p, E_R)$. In particular, $\operatorname{Hom}_R(R_p, E_R) \not\cong E_{R_p}$.

Example 3.2. ([3, p. 127]) Let R be a local domain such that the completion of R has a nonminimal prime contracting to (0) in R. Let Q be the field of fractions of R. Then $\operatorname{Hom}_R(Q, E_R)$ is not Artinian.

Example 3.3. Let R be a Noetherian domain which is not local. Let $m \neq n$ be maximal ideals of R. By a slight modification of the proof of [3, Lemma 4.1], one obtains that (0) is an associated prime of $\operatorname{Hom}_R(R_m, E_R(R/n))$, which is a direct summand of $\operatorname{Hom}_R(R_m, E_R)$. Hence, $\operatorname{Hom}_R(R_m, E_R) \not\cong E_{R_m}$.

We now show that the converse to part (a) of Theorem 1.1 does not hold in general. Let R be a domain and Q its field of fractions. Of course, Q is reflexive as a $Q = R_{(0)}$-module. But as the following theorem shows, Q is rarely a reflexive R-module.

Proposition 3.4. *Let R be a Noetherian domain and Q the field of fractions of R. Then Q is a reflexive R-module if and only if R is a complete local domain of dimension at most one.*

Proof. We first suppose R is a one-dimensional complete local domain with maximal ideal m. Let $E = E_R(R/m)$. By [4, Theorem 2.5], $\operatorname{Hom}_R(Q, E) \cong Q$. Since the evaluation map of the Matlis double dual is always injective, we obtain that $Q \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(Q, E), E)$ is an isomorphism.

Conversely, suppose Q is a reflexive R-module. By Theorem 2.1, R is a complete semilocal domain, hence local. It suffices to prove that $\dim R \leq 1$. Again by Theorem 2.1, there exists a finitely generated R-submodule N of Q such that Q/N is Artinian. Since $\operatorname{Ann}_R N = 0$, $\dim R = \dim N$. Thus, it suffices to prove that $H^i_m(N) = 0$ for $i \geq 2$. But this follows readily from the facts that $H^i_m(Q) = 0$ for all i and $H^i_m(Q/N) = 0$ for $i \geq 1$ (as Q/N is Artinian). $\square$

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