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ON PROPERTIES OF MATRIX $I^{(-1)}$ OF SINC METHODS

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Abstract. In this paper, we study the determinant and eigen properties of $I^{(-1)}$, an important Toeplitz matrix used in Sinc methods. Some Sinc method applications depend on the non-singularity of this matrix and on the location of its eigenvalues. Among the theorems we prove is that $I^{(-1)}$ is nonsingular. We also show that if λ is a pure imaginary eigenvalue of $I^{(-1)}$, then $|\lambda| > \frac{1}{\pi}$.

1. Introduction

Our goal in this paper is to study properties of an important Toeplitz matrix in the theory of Sinc indefinite integration and Sinc convolution. This matrix is denoted as $I^{(-1)}$. Actually, this is an infinite family of Toeplitz matrices, generated by an infinite sequence of diagonal coefficients; definitions are given below. Although much is known about the global and asymptotic behavior of the eigenvalues of such families in general, as noted in [4, p. 187], much less is known about the behavior of individual eigenvalues.

We are particularly interested in a conjecture of Frank Stenger: $I^{(-1)}$ is diagonalizable and its eigenvalues are located in the open right half plane (see [7]). Both properties are needed for the convergence of certain convolution algorithms and differential equation solvers based on Sinc theory developed in [8]. As noted in [8], it has been determined numerically that these properties hold for all such matrices of order at most 512. Although this is sufficient for most practical purposes, the general problem remains open. It follows rather easily from the definition of $I^{(-1)}$ that its eigenvalues lie in the *closed* right half-plane. One

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of the results we will prove in this paper is that if λ is a pure imaginary eigenvalue of $I^{(-1)}$, then $|\lambda| > \frac{1}{\pi}$. It follows that $I^{(-1)}$ is always an invertible matrix. Moreover, we will show that $I^{(-1)}$ is at least close to diagonalizable in the sense that it is a rank one update of a highly structured skew-symmetric matrix with simple eigenvalues.

We use the following notation:

(1) The square constant matrix whose elements are all equal to ω is denoted by by $[\omega]$.
(2) If (λ, x) is an eigenpair of the matrix A, then we write $\lambda \in \text{evals}(A)$ and $x \in \text{evecs}(A)$. Also, we use the notation $\lambda(A)$ to mean that λ is an eigenvalue of A.
(3) The transpose (Hermitian transpose) of the matrix A is denoted by A^t (A^*) and the tuple notation $x = (x_1, x_2, \ldots x_n)$ represents the column vector $[x_1, x_2, \ldots x_n]^t$ in $\mathbb{C}^n$.

Next, we introduce the sinc function, Sinc methods and $I^{(-1)}$.

Definition 1.1. The sinc function is defined as follows:

$$\text{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & \text{for } x \neq 0 \\ 1, \ \text{for } x = 0. \end{cases}$$

Sinc methods are a family of formulas based on the *sinc function* which provide accurate approximation of derivatives, definite and indefinite integrals and convolutions. These methods were developed extensively by Frank Stenger and his students (see [9].) Sinc methods can be used to numerically solve differential equations with boundary layers, integrals with infinite intervals or with singular integrands, and ordinary differential equations or partial differential equations that have coefficients with singularities. These are two numerical methods which depend on the sinc function defined above. For more information, see [5], [8], and [7]. For convolutions and integration, the following family of matrices is especially important.

Definition 1.2. $I^{(-1)}$ is the $n \times n$ matrix defined as follows:

$$I^{(-1)} = [\eta_{ij}]_{i,j=1}^{n}$$

where $\eta_{ij} = e_{i-j}$, $e_k = \dfrac{1}{2} + s_k$, and $s_k = \displaystyle\int_0^k \text{sinc}(x)dx$

Thus, $I^{(-1)}$ can be expressed in the form

$$I^{(-1)} = \left[\frac{1}{2}\right] + S,$$

where S is the following skew-symmetric Toeplitz matrix of dimension n:

$$S = \begin{bmatrix} 0 & -s_1 & -s_2 & \dots & -s_{n-1} \\ s_1 & 0 & -s_1 & \dots & -s_{n-2} \\ s_2 & s_1 & 0 & \dots & -s_{n-3} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & \dots & 0 \end{bmatrix}$$

Throughout this paper, S refers to the matrix defined above. Observe that S is a real skew-symmetric Toeplitz matrix. If we want to emphasize the dimension of S, we write S_n in place of S. Also note that the matrix $[\frac{1}{2}] = \frac{1}{2}uu^*$, with $u = (1, 1, \dots, 1)$, is a rank one matrix. This decomposition of $I^{(-1)}$ is key to our analysis of its behavior.

2. Localization Results for $I^{(-1)}$

We have verified computationally (for a dimension up to 1000) that the eigenvalues of $I^{(-1)}$ lie in the open right half plane. $I^{(-1)}$. The fact that in all cases they lie in the closed right half plane follows easily from the following elementary, but key result.

Lemma 2.1. *Let (λ, x) be an eigenpair of $A = [\omega] + M$, ω real and M skew-Hermitian, $x = (x_1, x_2, \dots x_n)$ and $\|x\|_2 = 1$. Then*

$$\mathrm{Re}(\lambda) = \frac{1}{2}\left|\sum_{j=1}^{n} x_j\right|^2 \textit{ and } \mathrm{Im}(\lambda) = x^* M x$$

Proof. With the given notation we have from $Ax = \lambda x$ that

$$\lambda = x^* \left[\omega\right] x + x^* M x.$$

However, $x^* \left[\omega\right] x = \omega \left|\sum_{j=1}^{n} x_j\right|^2$ is real and $x^* M x$ is pure imaginary since M is skew-Hermitian. The result follows. $\square$

In particular, it follows from this lemma that the real part of an eigenvalue λ of $I^{(-1)}$ is nonnegative. The lemma can also be used to further localize the eigenvalues of $I^{(-1)}$ as follows: the real part of each eigenvalue of $I^{(-1)}$ is between 0 and $n/2$. To see this, note that the eigenvalues of the rank 1 matrix $\left[\frac{1}{2}\right]$ are $n/2$ and 0 (with multiplicity $n-1$.) Since $\|x\|_2 = 1$, it follows that $x^* \left[\frac{1}{2}\right] x \le n/2$.

Thus the issue of eigenvalues of $I^{(-1)}$ lying in the open half right plane is reduced to showing that no eigenvalues of $I^{(-1)}$ lie on the imaginary axis. The main result of this section shows that if there is a purely imaginary eigenvalue $\lambda = bi$ of $I^{(-1)}$, then $|b| > 1/\pi$.

Definition 2.2. If $v = [v_0, v_1, ..., v_n]^t$ is an eigenvector of A, then the corresponding *eigenpolynomial* is

$$V(z) = v_0 + v_1 z + ... + v_n z^n.$$

A function $f(x)$ is said to be a *generator* for a matrix $M = [m_{ij}]$ of dimension n, if $m_{ij} = c_{i-j}$, for all $1 \leq i, j \leq n$, where

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx, \quad k = 0, 1, ..., n-1.$$

In the following lemma, we prove that S is the Toeplitz matrix generated by the complex function i/x. In what follows, the integral sign denotes the Cauchy Principal Value and not the Lebesgue integral which has been used by other authors in dealing with generators.

Lemma 2.3. $\frac{i}{x}$ *is a generator of S.*

Proof. We want $f(x) = \frac{i}{x}$ to satisfy:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-irx}dx = \int_0^r \frac{\sin(\pi x)}{\pi x}\,dx, \;\; r = 0, 1, ..., n-1.$$

Now we claim that

$$\frac{i}{2\pi}\int_{-\pi}^{\pi} \frac{\cos(rx) - i\sin(rx)}{x}dx = \int_0^r \frac{\sin(\pi x)}{\pi x}dx. \tag{2.1}$$

First notice that if $r = 0$, then the left hand side of (2.1) is $\frac{i}{2\pi}\int_{-\pi}^{\pi}\frac{dx}{x}$, which is equal to zero. And if $r \neq 0$, then $\frac{\cos(rx)}{x}$ is odd. So the left hand side of (2.1) is now

$$\frac{i}{2\pi}\int_{-\pi}^{\pi} -i\frac{\sin(rx)}{x}dx, \tag{2.2}$$

which is an ordinary integral since the integrand is continuous. In fact, the integrand is even. Thus, (2.2) becomes

$$\frac{1}{\pi}\int_0^{\pi} \frac{\sin(rx)}{x}dx \tag{2.3}$$

Now substitute $x = \frac{\pi}{r}y$ in (2.3), to get

$$\int_0^r \frac{\sin(\pi y)}{\pi y}dy,$$

which proves the lemma. $\square$

To study the determinant and eigen properties of S, it suffices to study the matrix $T \equiv -iS$ instead. T is generated by $1/x$. It is Hermitian and has the advantage of being generated by a real-valued function. These facts are helpful in the following theorem, the proof of which is modeled along the lines of Theorem 2.1 of [10].

Theorem 2.4. *If μ is a pure imaginary eigenvalue of $I^{(-1)}$, then $|\mu| > \frac{1}{\pi}$.*

Proof. Let (μ, u) be an eigenpair of $I^{(-1)}$, where $\mu = ib$, b is real, and without loss of generality, assume that b is nonnegative. It follows from the identity $\mathrm{I}^{(-1)}x = ibx$ that

$$x^* \left[\frac{1}{2}\right] x + x^* S x = ib \left\|x\right\|^2 .$$

Since the right hand side and x^*Sx are purely imaginary, it follows that $0 = x^* \left[\frac{1}{2}\right] x = \frac{1}{2}\left|\sum_{j=1}^{n} x_j\right|^2$. Therefore $\left[\frac{1}{2}\right] x = 0$, from which it follows that (η, x) is also an eigenpair of S and $(0, x)$ is an eigenpair of $[\frac{1}{2}]$. This means that (b, x) is an eigenpair of T. Now if $U(z)$ is the eigenpolynomial associated with the eigenvector u, then it follows that 1 is a zero of $U(z)$. For the fact that $(0, u)$ is an eigenpair of $[\frac{1}{2}]$ implies that the sum of the coordinates of u is zero, which means that the sum of the coefficients of $U(z)$ is zero. Thus, we can write $U(z) = (z-1)\hat{U}(z)$, for some polynomial $\hat{U}(z)$. Now let $V(z)$ be a polynomial corresponding to a vector v and which has -1 as a zero. Then $V(z)$ can be factored as $V(z) = (z+1)\hat{V}(z)$. Thus, $U(z)\overline{V(z)} = (z-1)\overline{(z+1)}\hat{U}(z)\overline{\hat{V}(z)}$. Now notice that if $z = e^{i\theta}$, then $(z-1)\overline{(z+1)} = 2i\sin(\theta)$. Since $f(\theta) = 1/\theta$ is a generator of T, it follows that

$$\begin{aligned} \frac{2i}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(\theta)}{\theta}\hat{U}(e^{i\theta})\overline{\hat{V}(e^{i\theta})}d\theta &= \langle Tu, v\rangle = \langle \lambda u, v\rangle \\ &= \frac{2i\lambda}{2\pi}\int_{-\pi}^{\pi} \sin(\theta)\hat{U}(e^{i\theta})\overline{\hat{V}(e^{i\theta})}d\theta. \end{aligned}$$

Thus,

$$\int_{-\pi}^{\pi} \left(\frac{\sin(\theta)}{\theta} - \lambda\sin(\theta)\right)\hat{U}(e^{i\theta})\overline{\hat{V}(e^{i\theta})}d\theta = 0.$$

Observe that $\hat{V}(z)$ can be chosen to be any polynomial of degree at most $n-1$. Make the choice $\hat{V}(z) = \hat{U}(z)$ to obtain

$$\int_{-\pi}^{\pi} \left(\frac{\sin(\theta)}{\theta} - b\sin(\theta)\right)|\hat{U}(e^{i\theta})|^2 d\theta = 0.$$

Since u is an eigenvalue, the polynomial $\hat{U}(z)$ is nonzero. It follows that the integrand above is positive on the open interval $(-\pi, 0)$. Moreover, the factor $\sin(\theta)|\hat{U}(e^{i\theta})|^2$ is also positive on the interval $(0, \pi)$. It follows that $\frac{1}{\theta}-b < 0$ for some θ in the interval $(0, \pi)$. This implies that $b > 1/\pi$ and completes the proof. □

It follows from Theorem 2.4 that zero is excluded as a possible eigenvalue of $I^{(-1)}$.

Corollary 2.5. *For all dimensions n the matrix $I^{(-1)}$ is nonsingular.*

3. Diagonalizability of S_n

The principal theorem of this section is an analogue to Theorem 2.2 of [10]. Trench's theorem is valid only for real-valued Lebesgue-integrable functions on $(-\pi,\pi)$, which are not constant on a set of measure 2π, so it cannot be applied to $f(x) = 1/x$ and hence, to the matrix generated by $-if(x)$. In the following theorem, we shall use the notation S_n instead of S to emphasize the dimension (which is n), and use T_n to represent $-iS_n$.

Theorem 3.1. *The eigenvalues of $-iS_n = T_n(t_{r-s})_{r,s=1}^n$ are simple, where*

$$t_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\theta} e^{-ik\theta} d\theta, \quad k = 0, 1, ..., n-1. \tag{3.1}$$

Proof. Let $f(x) = 1/x$. First, note that $f(x)$ is a generator of T_n. We will show that if λ_r is an eigenvalue of T_n of multiplicity greater than or equal to 2, then $f(x) - \lambda_r$ must change sign at least 3 times in $(-\pi, \pi)$. Clearly $\dfrac{1}{x} - \lambda_r$ does not change sign more than once in $(-\pi, \pi)$ for any choice of λ_r. This will imply that all eigenvalues of T_n are simple.

Note that all of the following integrals are finite, because the Cauchy principal part of any integral whose integrand is a product of a polynomial and $1/\theta$ is finite.

Now for each vector $v = [v_1, v_2, ..., v_n]^t$ in $\mathbb{C}^n$, associate the polynomial

$$V(z) = [1, z, z^2, ..., z^{n-1}]v = \sum_{j=1}^{n} v_j z^{j-1}.$$

If u and v are in $\mathbb{C}^n$, then the standard inner product on $\mathbb{C}^n$ satisfies

$$\langle u, v\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(z)\overline{V(z)} d\theta, \tag{3.2}$$

where $z = e^{i\theta}$.

Also from (3.1), we obtain that

$$\langle T_n u, v\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\theta}U(z)\overline{V(z)}d\theta. \tag{3.3}$$

Now let $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be the eigenvalues of the Hermitian matrix T_n, with corresponding eigenvectors $x_1, x_2, ..., x_n$, and corresponding eigenpolynomials

$$X_i(z) = [1, z, ..., z^{n-1}]x_i, \quad 1 \leq i \leq n.$$

From (3.2) we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_i(z)\overline{X_j(z)}d\theta = \delta_{ij}, \quad 1 \leq i, j \leq n. \tag{3.4}$$

From 3.3 we also have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\theta}X_i(z)\overline{X_j(z)}d\theta = \delta_{ij}\lambda_i, \quad 1 \leq i, j \leq n, \tag{3.5}$$

where δ_{ij} is the Kronecker delta. Thus, if λ_r is an eigenvalue of multiplicity one, then we claim that $f(\theta) - \lambda_r$ must change sign at some point. This point must be $1/\lambda_r$ (if $\lambda_r \neq 0$) or 0 (if $\lambda_r = 0$). Notice that multiplying (3.4) by λ_r, and taking $i = j = r$, yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \lambda_r |X_r(z)|^2 d\theta = \lambda_r. \tag{3.6}$$

And by taking $i = j = r$ in (3.5), we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\theta}|X_r(z)|^2 d\theta = \lambda_r. \tag{3.7}$$

Now subtract (3.6) from (3.7) to obtain that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\theta} - \lambda_r\right)|X_r(z)|^2 d\theta = 0.$$

Now if $f(\theta) - \lambda_r$ is of constant sign in $(-\pi, \pi)$, say it is positive, then the integral

$$\int_{-\pi}^{\pi}\left(\frac{1}{\theta} - \lambda_r\right)|X_r(z)|^2 d\theta$$

should be positive. This is impossible, which proves our assertion about the sign change of $f(\theta) - \lambda_r$.

Next suppose that $m \geq 2$. The expression $(1 - \theta\lambda_r)/\theta$ changes sign only at θ_1 and θ_2, where $\theta_1 = 0$ and $\theta_2 = \lambda_r$. We will show that this assumption (that $m \geq 2$) leads to a contradiction. Notice that it suffices to handle the case when $\lambda_r \geq 0$, since nonzero eigenvalues of S_n are of the form ia, $a \neq 0$, $a \in \mathbb{R}$, and they occur in pairs. Therefore, if $-a$ (assume $a > 0$) is an eigenvalue of T_n of multiplicity greater than or equal to 2, then a is an eigenvalue of multiplicity greater than or equal to 2.

So assume that $\lambda_r \geq 0$, and define

$$g(\theta) = \frac{1}{2\pi}\left(f(\theta) - \lambda_r\right). \tag{3.8}$$

We assert that the function $h(\theta)$ defined by

$$h(\theta) = g(\theta)\sin\left(\frac{\theta - \theta_1}{2}\right)\sin\left(\frac{\theta - \theta_2}{2}\right) \tag{3.9}$$

does not change sign in $(-\pi, \pi)$.

To see why this is true, notice first that since θ and θ_i, $i = 1, 2$, are all in $(-\pi, \pi)$, then $\frac{\theta-\theta_i}{2}$ is also in $(-\pi, \pi)$. Thus, $\sin(\frac{\theta-\theta_i}{2})$ defined on $(-\pi, \pi)$ can change sign only when $\frac{\theta-\theta_i}{2} = 0$; i.e. when $\theta = \theta_i$, for $i = 1, 2$. But, $g(\theta)$ changes sign only at these points. Consequently, $h(\theta)$ can change sign only at these points also. To show that h does not change sign at these two points, it suffices to prove that the sign of $h(\theta)$ does not change in a small neighborhood of θ_i, for $i = 1, 2$. Notice that for each i, we can choose this neighborhood not to include θ_j, for $j \neq i$.

Now write h as

$$h(\theta) = \frac{1}{2\pi}h_1(\theta)h_2(\theta).$$

where

$$h_1(\theta) = \frac{\sin(\theta/2)}{\theta}\text{and}h_2(\theta) = (1 - \theta\lambda_r)\left(\sin(\frac{\theta - \theta_2}{2})\right).$$

Recall that $\theta_1 = 0$ and $\theta_2 = \lambda_r$. First, let check the sign of h about θ_1. But, h_1 does not change sign at θ_1, nor does h_2. Thus, h does not change sign at θ_1. It remains to check the sign of h about θ_2. Since h_1 does not change sign at θ_2, we need only to check the sign of h_2 about θ_2. But, when $\theta < \theta_2$, we have $1 - \theta\lambda_r > 0$ and $\sin((\theta - \theta_2)/2) < 0$. Thus when $\theta < \theta_2$, $h(\theta)$ is negative. On the other hand, when $\theta > \theta_2$, $1 - \theta\lambda_r < 0$, and $\sin((\theta - \theta_2)/2) > 0$. Thus, when $\theta > \theta_2$, $h(\theta)$ is negative. Therefore, h does not change sign at θ_2, and hence h does not change sign in $(-\pi, \pi)$.

Now suppose that $\lambda_r = \lambda_{r+1}$ is of multiplicity greater than or equal to 2. For $i = r, r+1$ and $1 \leq j \leq n$, we have that $\int_{-\pi}^{\pi} g(\theta)X_i(z)\overline{X_j(z)}d\theta =$

0, because

$$\begin{aligned}\int_{-\pi}^{\pi} g(\theta)X_i(z)\overline{X_j(z)}d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)X_i(z)\overline{X_j(z)}d\theta - \frac{\lambda_r}{2\pi}\int_{-\pi}^{\pi} X_i(z)\overline{X_j(z)}d\theta \\ &= \lambda_i\delta_{ij} - \lambda_r\delta_{ij} \\ &= (\lambda_i - \lambda_r)\delta_{ij}.\end{aligned}$$

Now if $i = r$, then the above integral is zero. It is also zero if $i = r+1$, because $\lambda_{r+1} = \lambda_r$. Therefore,

$$\int_{-\pi}^{\pi} g(\theta)p(z)\overline{X_j(z)}d\theta = 0, \quad 1 \le j \le n,$$

where

$$p(z) = c_0X_r(z) + c_1X_{r+1}(z)$$

and c_0 and c_1 are constants, which are not both zero, and which satisfy

$$c_0X_r(1) + c_1X_{r+1}(1) = 0.$$

This means that $p(z)$ is a nonzero polynomial.

Now set $Q(z) = \dfrac{p(z)(z - e^{i\theta_2})}{z-1}$, which is a polynomial of degree less than or equal to $n-1$, and obtain that

$$\int_{-\pi}^{\pi} g(\theta)p(z)\overline{Q(z)}d\theta = 0. \tag{3.10}$$

Thus,

$$\int_{-\pi}^{\pi} g(\theta)|p(z)|^2\left(\frac{\overline{z} - e^{-i\theta_2}}{\overline{z} - 1}\right)d\theta = 0.$$

If we let

$$g_1(\theta) = g(\theta)\left|\frac{\sum_{l=0}^{1} c_lX_{r+l}(z)}{z-1}\right|^2,$$

then the above equation becomes

$$\int_{-\pi}^{\pi} g_1(\theta)(z-1)(\overline{z} - e^{-i\theta_2})d\theta = 0. \tag{3.11}$$

Now if $z = e^{i\theta}$, then

$$(z-1)(\overline{z} - e^{-i\theta_2}) = 4e^{-i\frac{\theta_2}{2}}\sin(\frac{\theta}{2})\sin(\frac{\theta-\theta_2}{2}).$$

Notice, that

$$e^{i\theta}e^{i\phi} = e^{i(\frac{\theta+\phi}{2})}(e^{i(\frac{\theta-\phi}{2})} - e^{-i(\frac{\theta-\phi}{2})} = 2e^{i(\frac{\theta+\phi}{2})}\sin(\theta - \phi).$$

Therefore, (3.11) becomes

$$4\int_{-\pi}^{\pi} g_1(\theta)\left(e^{-i\frac{\theta_2}{2}}\sin(\frac{\theta}{2})\sin(\frac{\theta-\theta_2}{2})\right)d\theta = 0. \tag{3.12}$$

Thus,

$$\int_{-\pi}^{\pi} g_1(\theta)\sin(\frac{\theta}{2})\sin(\frac{\theta-\theta_2}{2})d\theta = 0. \tag{3.13}$$

This implies that

$$\int_{-\pi}^{\pi} \alpha(z)g(\theta)\left(\sin(\frac{\theta}{2})\sin(\frac{\theta-\theta_2}{2})\right)d\theta = 0.$$

where

$$\alpha(z) = \left|\frac{p(z)}{z-1}\right|^2.$$

Therefore,

$$\int_{-\pi}^{\pi} \alpha(z)h(\theta)d\theta = 0.$$

Since h does not change sign in $(-\pi,\pi)$ and $\alpha(z)\geq 0$, then it must be that $p(z)/(z-1)\equiv 0$. Therefore, $p\equiv 0$. This is a contradiction which proves that the multiplicity of any eigenvalue is 1. ☐

The equalities $S=T$ and $I^{(-1)} = [\frac{1}{2}]+S$ show that $I^{(-1)}$ is at most a rank one update away from a matrix with simple eigenvalues, as stated in the introduction.

Corollary 3.2. *The matrix $I^{(-1)}$ can be expressed as the sum of a rank one matrix and a skew-symmetric matrix with simple eigenvalues.*

We conclude with a result relating the determinants of the matrices S and $I^{(-1)}$. It depends on a fact that can be given a more general context than sinc matrices, and we do so.

Theorem 3.3. *Let M be a non-singular real skew-Hermitian matrix and $A=[\omega]+M$, where ω is a nonzero real number. Then* $\det(A)=\det(M)$.

Proof. The rank one matrix $M^{-1}[\omega]$ has at most one nonzero eigenvalue. This eigenvalue must be real, since eigenvalues of a real matrix occur in conjugate pairs. Consequently, all eigenvalues of $M^{-1}[\omega]$ are real. Let (λ,x) be an eigenpair for this matrix. From $M^{-1}[\omega]x=\lambda x$ we deduce that $x^*[\omega]x=\lambda x^*Mx$. The right hand side of this identity is pure imaginary and the left hand side real, so each is zero. It follows that $[\omega]x=0$, so that $0=[\omega]x=\lambda Mx$. Since M is non-singular, it follows that $\lambda = 0$. Therefore all eigenvalues of $M^{-1}[\omega]$ are zero. Consequently the only eigenvalues of $I + M^{-1}[\omega]$ are 1. Now we calculate

$$\det(A) = \det(M(I + M^{-1}[\omega])) = \det(M) \cdot 1$$

$\square$

Corollary 3.4. *Let n be even. Then* $\det(I_n^{(-1)}) = \det(S_n)$.

Proof. Since S_n is skew-symmetric and even, all eigenvalues occur in signed pairs $\pm ib$, where b is real. Therefore, if zero were an eigenvalue of S_n, it would be a repeated eigenvalue. Such eigenvalues do not occur by Theorem 3.1. Therefore, S_n is nonsingular and Theorem 3.3 applied to the equality $I^{(-1)} = [\frac{1}{2}] + S$ gives the desired result. $\square$

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