

ASSIGNMENT 3 KEY FOR CSCE/MATH 441

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Points: 40  
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**Exercise 5.5.** By following the procedure described in Section 5.5, construct a divided difference table for the polynomial in  $\mathcal{P}_4$  that interpolates the function values  $f(0), f'(0), f''(0), f(1), f'(1)$ . Test your calculation on the polynomial function  $f(x) = (1+x)^4$ ,  $0 \leq x \leq 1$ .

SOLUTION. (8) Here is the difference table for a Hermite quartic polynomial interpolating  $f(x)$  at  $0, 0, 0, 1, 1$ . We use the formula of text (5.27) that says  $f[x_j, x_{j+1}, \dots, x_{j+k}] = f^{(k)}(x_j)/k!$  for  $x_{j+i} = x_j$ .

$x_j$	$f[x_j]$	$f[x_j, x_{j+1}]$	$f[x_j, x_{j+1}, x_{j+2}]$	$f[x_j, x_{j+1}, x_{j+2}, x_{j+3}]$	$f[x_j, x_{j+1}, x_{j+2}, x_{j+3}, x_{j+4}]$
0	$f(0)$				
		$f'(0)$			
0	$f(0)$		$\frac{f''(0)}{2}$		
		$f'(0)$		$f[0, 1] - f'(0) - \frac{f''(0)}{2}$	
0	$f(0)$		$f[0, 1] - f'(0)$		$f'(1) - 3f[0, 1] + 2f'(0) + \frac{f''(0)}{2}$
		$f[0, 1]$		$f'(1) - 2f[0, 1] + f'(0)$	
1	$f(1)$		$f'(1) - f[0, 1]$		
		$f'(1)$			
1	$f(1)$				

Now test this with

$$\begin{aligned}
 f(x) &= (1+x)^4 \\
 f'(x) &= 4(1+x)^3 \\
 f''(x) &= 12(1+x)^2
 \end{aligned}$$

so the resulting polynomial is

$$\begin{aligned}
 p(x) &= 1 + 4(x-0) + 12(x-0)^2 + \left(2^4 - 1 - 4 - \frac{12}{2}\right)(x-0)^3 \\
 &\quad + \left(4 \cdot 2^3 - 3(2^4 - 1) + 2 \cdot 4 + \frac{12}{2}\right)(x-0)^3(x-1) \\
 &= 1 + 4x + 12x^2 + 5x^3 + x^3(x-1) \\
 &= 1 + 4x + 12x^2 + 4x^3 + x^4 = (1+x)^4.
 \end{aligned}$$

**Exercise 5.8.** You are given a dataset of function values. Detect and fix two errors as best you can.

SOLUTION. (8) It's instructive to make out a divided difference table and map out the sign changes which can be done by graphing columns of the table. Now look for the pattern

I suggested that would result from a single error of  $\epsilon$  in one entry and zeros in the rest. These suggest an error in the 5th place. Make a correction of +0.003 and continue graphing. It appears that there is an error in the 9th place. Make a correction of +0.000009 and the table looks good to about the 4th column.

Here were a few of the commands in the experiment:

```
load 'ord5_8' % load up data vector y
z = Dif(y);
n = 2, plot(z(1:12-n,n)),grid
y(5) = y(5)+0.0005;
diffs = Dif(y);
n = 2, plot(z(1:12-n,n)),grid
```

**Exercise 6.2.** By using the identity

$$k^2 = (k-1)(k-2) + 3(k-1) + 1,$$

prove that the Bernstein approximation to the function  $f(x) = x^3$ ,  $0 \leq x \leq 1$ , is the polynomial

$$p(x) = \frac{(n-1)(n-2)}{n^2}x^3 + \frac{3(n-1)}{n^2}x^2 + \frac{1}{n^2}x, \quad 0 \leq x \leq 1.$$

SOLUTION. (8) We have

$$\begin{aligned} (B_n f)(x) &= \sum_{k=0}^n \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \left(\frac{k}{n}\right)^3 \\ &= \frac{1}{n^2} \sum_{k=0}^n \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \left(\frac{k}{n}\right) ((k-1)(k-2) + 3(k-1) + 1) \end{aligned}$$

Not treat each of these three sums separately:

$$\sum_{k=0}^n \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \left(\frac{k}{n}\right) \cdot 1 = x,$$

as shown in the text, proof of Theorem 6.3, with the key observation that the term with  $k=0$  vanishes and a change of variable. Next note that the first two terms of the next sum vanish, so make the change of variable  $j = k-2$  to obtain

$$\begin{aligned} &\sum_{k=0}^n \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \left(\frac{k}{n}\right) (k-1)(k-2) \\ &= 3 \sum_{k=0}^n \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \left(\frac{k}{n}\right) (k-1) \\ &= 3(n-1)x^2 \sum_{k=2}^n \frac{(n-2)!}{(k-2)!(n-k)!} x^{k-2} (1-x)^{n-k} \\ &= 3(n-1)x^2 \sum_{j=0}^{n-2} \frac{(n-2)!}{j!(n-2-j)!} x^{k-2} (1-x)^{n-2-j} = 3(n-1)x^2 \cdot 1 \end{aligned}$$

Finally, note that the first three terms of the last sum vanish, and do a change of variables  $j = k - 3$  to obtain

$$\begin{aligned} & \sum_{k=0}^n \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \binom{k}{n} (k-1)(k-2) \\ &= \sum_{k=0}^n \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \binom{k}{n} (k-1)(k-2) \\ &= (n-1)(n-2) x^3 \sum_{k=3}^n \frac{(n-3)!}{(k-3)!(n-k)!} x^{k-3} (1-x)^{n-k} \\ &= (n-1)(n-2) x^3 \sum_{j=0}^{n-3} \frac{(n-3)!}{j!(n-3-j)!} x^{k-3} (1-x)^{n-3-j} = (n-1)(n-2) x^2 \cdot 1 \end{aligned}$$

**Exercise N3.** Write a Matlab function with the calling form `HCpp(x,f,fp)`, where this function returns a PP structure representing the Hermite cubic p.p. with node row vector `x`, corresponding function values in the row vector `f` and derivative values in the row vector `fp`. You should follow the guidelines of the file `PPfcns.m` that is found in Course Materials and use the formula that we derived in class for Hermite cubics. Validate with some calculations with the Runge's example and several choices of knots and include printout for the case of three knots.

SOLUTION. (8) Here is the difference table for a Hermite cubic polynomial interpolating  $f(x)$  at  $a, a, b, b$ :

$x_j$	$f[x_j]$	$f[x_j, x_{j+1}]$	$f[x_j, x_{j+1}, x_{j+2}]$	$f[x_j, x_{j+1}, x_{j+2}, x_{j+3}]$
$a$	$f(a)$			
		$f'(a)$		
$a$	$f(a)$		$\frac{f[a,b] - f'(a)}{b-a}$	
		$f[a,b]$		$\frac{f'(b) + f'(a) - 2f[a,b]}{(b-a)^2}$
$b$	$f(b)$		$\frac{f'(b) - f[a,b]}{b-a}$	
		$f'(b)$		
$b$	$f(b)$			

Now we see that the interpolating polynomial is

$$\begin{aligned} p(x) &= f(a) + f'(a)(x-a) + \frac{f[a,b] - f'(a)}{b-a} (x-a)^2 + \frac{f'(b) + f'(a) - 2f[a,b]}{(b-a)^2} (x-a)^2 (x-b) \\ &= f(a) + f'(a)(x-a) + \frac{f[a,b] - f'(a)}{b-a} (x-a)^2 + \frac{f'(b) + f'(a) - 2f[a,b]}{(b-a)^2} (x-a)^2 (x-a+a-b) \\ &= f(a) + f'(a)(x-a) + \left( \frac{3f[a,b] - 2f'(a) - f'(b)}{b-a} \right) (x-a)^2 + \frac{f'(b) + f'(a) - 2f[a,b]}{(b-a)^2} (x-a)^3 \end{aligned}$$

The only change in the code of `HCpp.m` is to change the code in the for loop to

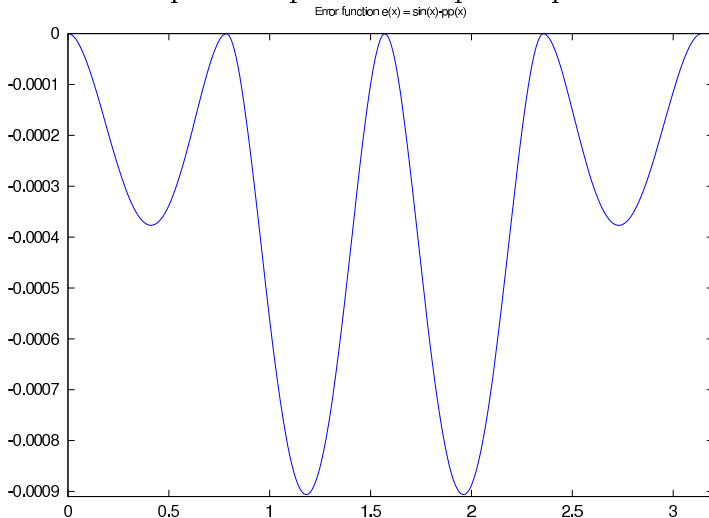
```
% use differences to determine the coefficients
```

```
h = x(j+1) - x(j);
```

```
fab = (f(j+1)-f(j))/h;
pp.coefs(j,:) = [(fp(j)+fp(j+1)-2*fab)/(h*h), (3*fab-2*fp(j)-fp(j+1))/h, fp(j),
f(j)];
```

A sample run in Octave on  $f(x) = \sin x$ ,  $0 \leq x \leq \pi$ , is as follows:

```
octave:2> PPfens
octave:3> xnodes = linspace(0,pi,5);
octave:4> f = sin(xnodes);
octave:5> fp = cos(xnodes);
octave:6> pp = HCpp(xnodes,f,fp) pp =
{
order = 4
knum = 5
knots =
0.00000 0.78540 1.57080 2.35619 3.14159
coefs =
-0.15162 -0.00784 1.00000 0.00000
-0.06280 -0.37617 0.70711 0.70711
 0.06280 -0.52415 0.00000 1.00000
 0.15162 -0.36508 -0.70711 0.70711
}
octave:7> x = 0:.01:pi;
octave:8> plot(x,PPeval(pp,x)-sin(x))
octave:9> title('Error function e(x) = sin(x)-pp(x)')
octave:10> print -deps ErrorGraphN3.eps
```



**Exercise N4.** It is claimed in the notes that the error of Hermite p.p. interpolation is  $\mathcal{O}(h^4)$ , where  $h$  is the largest distance between knots, so that, if the error estimate is reasonably sharp, halving  $h$  should reduce the error by a factor of about 16. Verify this with Runge's example on the interval  $[0, 5]$  and equally spaced knots with  $h = 1$ ,  $h = 1/2$  and  $h = 1/4$ .

SOLUTION. (8) In order to do this, we assume that the error is well approximated as  $Ch^4$ . This, when we halve the step size, we expect the error to reduce by a factor of 16. Here is the Octave script we used:

```
% script: N4.m
% description: script for exercise N4
PPfcns % load up the PP library
knotnum = [6,11,21]; % knot numbers for h = 1,1/2,1/4
x = 0:.0001:5; % grid points for error evaluation
err = zeros(size(knotnum)); % error vector
for j = 1:3
    knots = linspace(0,5,knotnum(j));
    f = 1./(1+knots.^2); % function value at knots
    fp = -2*knots./(1+knots.^2).^2; % function derivative value at knots
    pp = HCpp(knots,f,fp); % generate Hermite cubic p.p.
    err(j) = norm(1./(1+x.^2)-PPEval(pp,x),inf); % calculate error at x
end
% display the ratios
[err(1)/err(2),err(2)/err(3)]
The resulting output:
ans =
10.3350 6.7122
Good, but not as good as the proposed factor of 16.
```