Math 314 Lecture Notes
Section 006
Fall 2006

CHAPTER 1

Linear Systems of Equations

First Day:

(1) Welcome
(2) Pass out information sheets
(3) Take roll
(4) Open up home page and have students do same to check for login problems
(5) Go through information materials

1. Some Examples

(8/22/06) We examine the key definitions:

DEFINITION 1.1. A *linear equation* in the variables $x_1, x_2, ..., x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

where the coefficients $a_1, a_2, ..., a_n$ and right hand side constant term b are given constants.

DEFINITION 1.2. A *linear system* of m equations in the n unknowns $x_1, x_2, ..., x_n$ is a list of m equations of the form

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots a_{in}x_n & = & b_i \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mj}x_j + \cdots a_{mn}x_n & = & b_m. \end{array} \tag{1}$$

Then we look at specific examples and discuss specific examples

- $$\begin{array}{rcl} x + 2y & = & 5 \\ 4x + y & = & 6 \end{array}$$
- $$\begin{array}{rcl} x + y + z & = & 4 \\ 2x + 2y + 5z & = & 11 \\ 4x + 6y + 8z & = & 24 \end{array}$$

and the implications of these examples.

Next, for a more general application example, we look at a closed Leontief input-output model from the section.

2. Notation and a Review of Numbers

(8/24/06) We review some set notation that we'll use throughout the course. For example,

$a \in A$ means "a is a member of the set A."
$A = B$ means "the set A is equal to the set B."
$A \subseteq B$ means "A is a subset of B."
$A \subset B$ means "A is a proper subset of B."

Also the set by rule notation $A = \{x \mid x \text{ satisfies } ...\}$

Next a review of number notation and a detailed analysis of complex arithmetic, with examples of complex arithmetic and standard form.

State the fundamental theorem of algebra and develop further tools such as polar notation to solve polynomial equations.

3. Gaussian Elimination: Basic Ideas

(8/29/06) Introduce students to formal definitions of matrices and vectors, along with elementary row operations, by way of this simple example:

$$\begin{aligned} 2x - y &= 1 \\ 4x + 4y &= 20. \end{aligned} \tag{2}$$

Solution is $x = 2$, $y = 3$.

Definition 3.1. A *matrix* is a rectangular array of numbers. If a matrix has m rows and n columns, then the *size* of the matrix is said to be $m \times n$. If the matrix is $1 \times n$ or $m \times 1$, it is called a *vector*. If $m = n$, then it is called a *square matrix of order* n. Finally, the number that occurs in the ith row and jth column is called the (i, j)th *entry* of the matrix.

Introduce the standard notation. The statement "$A = [a_{ij}]$" means that A is a matrix whose (i, j)th entry is denoted by a_{ij}. Generally, the size of A will be clear from context. If we want to indicate that A is an $m \times n$ matrix, we write

$$A = [a_{ij}]_{m,n} .$$

Similarly, the statement "$\mathbf{b} = [b_i]$" means that b is a n-vector whose ith entry is denoted by b_i. In case the type of the vector (row or column) is not clear from context, the default is a column vector. Many of the matrices we encounter will be *square*, that is, $n \times n$. In this case we say that n is the *order* of the matrix. Another term that we will use frequently is the following.

Definition 3.2. The *leading entry* of a row vector is the first nonzero element of that vector. If all entries are zero, the vector has no leading entry.

Describe the matrices of the general linear system of Equation (1.1) First, there is the $m \times n$ coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Notice that the way we subscripted entries of this matrix is really very descriptive: the first index indicates the row position of the entry and the second, the column position of the entry. Next, there is the $m \times 1$ right hand side vector of constants

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix}.$$

Finally, stack this matrix and vector along side each other (we use a vertical bar below to separate the two symbols) to obtain the $m \times (n+1)$ augmented matrix

$$\widetilde{A} = [A \mid \mathbf{b}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} & b_i \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} & b_m \end{bmatrix}.$$

Discuss the elementary row operations:

- E_{ij} : This is shorthand for the elementary operation of *switching the ith and jth rows* of the matrix.
- $E_i(c)$: This is shorthand for the elementary operation of *multiplying the ith row by the nonzero constant c.*
- $E_{ij}(d)$: This is shorthand for the elementary operation of *adding d times the jth row to the ith row.* (Read the symbols from right to left to get the right order.)

Discuss the difference between Gaussian elimination and Gauss-Jordan elimination. Introduce the pivot idea – entry used to clear out entries above and/or below it.

Finally, discuss systems with non-unique solutions, systems with no solutions, consistent versus inconsistent, and the difference between free and bound variables.

4. Gaussian Elimination: General Procedure

(8/31/06) Introduce the idea of a solution set of a system. Do the text example of a nonlinear system that shows irreversibility of certain algebraic operations.

DEFINITION 4.1. A *solution vector* for th general linear system given by Equation (1.1) is a vector

$$\mathbf{x} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

such that the resulting equations are satisfied for these choices of the variables. The set of all such solutions is called the *solution set* of the linear system, and two linear systems are said to be *equivalent* if they have the same solution sets.

Highlight reversibility of elementary operations on linear systems by exhibiting inverses and showing solution sets do not change under the operations.

Introduce the RREF:

DEFINITION 4.2. A matrix R is said to be in *reduced row form* if:

- **(1):** The nonzero rows of R precede the zero rows.
- **(2):** The column numbers of the leading entries of the nonzero rows, say rows $1, 2, ..., r$, form an increasing sequence of numbers $c_1 < c_2 < \cdots < c_r$.

The matrix R said to be in *reduced row echelon form* if, in addition to the above:

- **(3):** Each leading entry is a 1.
- **(4):** Each leading entry has only zeros above it.

Discuss uniqueness and do examples.

Conclude this section with discussion of rank, nullity and connection to system solving.

Introduction to Maple

Everyone should log into their account. Next look for the Maple program and start it up – Maple 10, not the classic view. A window will open up with a side bar of options. Close the Quick Help window and Tip menu. Now we're ready to go.

- Click on the Help menu, then Maple Help (notice, BTW, there is a Maple Tour – take it sometime at your leisure.)
- Type "solve" in the search window and press Search.
- Download from our home directory the Maple worksheet "MapleWorksheet#1.mw".
- Now click on File -> Open and find the worksheet, then open it.
- Read and follow instructions in this worksheet.

CHAPTER 2

Matrix Algebra

1. Matrix Addition and Scalar Multiplication

(9/3/06) We follow this section fairly closely to the text in developing the basic properties and examples of matrix addition and scalar multiplication. A key idea here is that of a linear combination of vectors or matrices.

2. Matrix Multiplication

(9/5/06)Again, we follow this section fairly closely to the text. A key idea is that we can express a linear system as a matrix-vector product. A few things were skipped. In particular, you might have a look at Example 2.10 before you attempt Exercise 9.

3. Applications of Matrix Arithmetic

(9/12/06) We cover some of these examples rather lightly, leaving details for the reader to flesh out in the text.

(9/14/06) We also do a Maple tutorial at the end of the section, which is in the notebook MapleWorksheet#2.mw to be found in our home directory. You'll need this material for homework assignment #1.

4. Special Matrices and Transposes

(9/19/06) We will cover most of this section, which is largely terminology. The most interesting new item here is that the rank of a matrix is unchanged if it is transposed.

5. Matrix Inverses

(9/21/06) We cover most of this section too. This is the demarcation point for Hour Exam 1, scheduled for 9/25/06.

6. Determinants

(9/26/06) We cover the section on determinants, mainly focusing on their properties and applications to inverses and Cramer's rule.

CHAPTER 3

Vector Spaces

1. Vector Spaces: Definitions and Basic Concepts

(9/28/06) We cover the basic ideas of Section 3.1 of the text: definitions, examples, linear operators.

2. Subspaces

(10/3/06) Definitions and the prime example of a subspace, the span of a set of vectors.

3. Linear Combinations

(10/10/06) We discuss properties of linear combinations including linear dependence/independence, connections to redundant vectors, basis and coordinates of a vector with respect to a basis. We examined the change of basis matrix that is a rotation matrix.

4. Subspaces Associated with Matrices and Operators

(10/12/06) We defined the null space, row space and column space of a matrix, along with the domain, range, target and kernel of a linear operator. Isomorphisms (1-1 onto linear operators) are discussed.

5. Bases and Dimension

(10/19/06) We discuss the key theorems: basis theorem (every finite dimensional vector space has a basis), dimension theorem (any two bases have same number of elements), connected by way of the handy Steinitz substitution theorem.