Sample Final Key **Math 221H** **Fall 2008**

Name:________________________ **Score:**__________

Instructions: Show your work in the spaces provided below for full credit. Clearly identify answers and show supporting work to receive any credit. Give exact answers (e.g., π) rather than inexact (e.g., 3.14); make obvious simplifications, e.g., 0 rather than $\sin\pi$. Point values are in parentheses. Notes, text or electronic equipment not allowed. Table entries for Laplace transforms may be used freely, unless otherwise stated. However, you must show enough detail so that it is clear that you are using table entries.

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(12) **1**. Classify each of the following systems as linear (L) or nonlinear (NL), autonomous(A) or non-autonomous (NA), and separable (S) or non-separable (NS).

(a) $xy' = y + e^y$ Solution: NL, NA, S

(b) $y\dfrac{dy}{dx} = xy - 1$ Solution: NL, NA, NS

(c) $xy' = xy + 1$ L, Solution: L, NA, NS

(d) $\frac{dx}{dt} = x + yx$, $\frac{dy}{dt} = -x + y$ Solution: NL, A, NS

(14) **2**. (a) Solve the IVP $\frac{dy}{dx} + 2xy = 0$, $y(0) = 1$.
Solution: Separate variables, giving $\frac{dy}{dx} = -2xy$, so $\frac{dy}{y} = -2xdx$.
Integrating gives $\ln|y| = -2\frac{x^2}{2} + C$, so that $y = \pm e^C e^{-x^2} = Ae^{-x^2}$.
Evaluate at $x = 0$ and obtain $y(0) = 1 = Ae^0 = A$ and solution is

$$y = e^{-x^2}.$$

(b) Solve the DE $y' - 2xy = e^{x^2}$.
Solution: This equation is linear, so integrating factor is $e^{\int -2xdx} = e^{-x^2}$.
Multiply both sides by e^{-x^2} and obtain

$$e^{-x^2}y' - 2xe^{-x^2}y = \left(e^{-x^2}y\right)' = e^{x^2}e^{-x^2} = 1.$$

Integrate both sides and obtain $e^{-x^2}y = x + C$, so that the solution is

$$y = xe^{x^2} + Ce^{x^2} = e^{x^2}(x + C).$$

(12) **3**. Use Euler's Method with a stepsize of $h = 1/3$ to the IVP $\frac{dy}{dx} = xy$, $y(0) = 1$ on the interval $[0, 1]$ to obtain an approximation to $y(1)$.
Solution: Here $x_0 = 0$, $y_0 = 1$, $x_1 = 1/3$, $x_2 = 2/3$, and $x_3 = 1$. The recursion formula comes from
$\frac{\Delta y}{\Delta x} = \frac{y_{n+1} - y_n}{h} \approx x_n y_n$, where $y_n \approx y(x_n)$, so that $y_3 \approx y(1)$.
so that $y_{n+1} = y_n + hx_n y_n = y_n\left(1 + \frac{1}{3}x_n\right)$.
Hence $y_1 = y_0\left(1 + \frac{1}{3}x_0\right) = 1\left(1 + \frac{1}{3}0\right) = 1$.
$y_2 = y_1\left(1 + \frac{1}{3}x_1\right) = 1\left(1 + \frac{1}{3}\frac{1}{3}\right) = \frac{10}{9} \approx 1.111$.
$y_3 = y_2\left(1 + \frac{1}{3}x_2\right) = \frac{10}{9}\left(1 + \frac{1}{3}\frac{2}{3}\right) = \frac{10}{9} \cdot \frac{11}{9} = \frac{110}{81} \approx 1.3580$.

(40) **4**. (a) Find two solutions to the homogeneous problem $x'' - 2x' + 2x = 0$.
SOLUTION: Characteristic equation is $r^2 - 2r + 2 = 0$, with roots

$$r = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 2}}{2} = 1 \pm i.$$

Therefore, two (independent) solutions to the equation are $x_1 = e^t \cos t$ and $x_2 = e^t \sin t$.

(b) Use (a) and the method of undetermined coefficients to find the general solution to the DE $x'' - 2x' + 2x = t + 1$.
SOLUTION: The family of derivatives of t and 1 is encompassed by t and 1 themselves, and these are not solutions to the homogeneous equation. So assume a particular solution $x_p = At + B$. Obtain that

$$x_p'' - 2x_p' + 2x_p = 0 - 2A + 2(At + B) = t + 1.$$

Collect terms and match coefficients to obtain that $2A = 1$ and $2B - 2A = 1$, so that $A = 1/2$ and $B = (1 + 2A)/2 = 1$. So
$x_p = \frac{1}{2}t + 1$. The general solution to the homogeneous equation is $x_h = C_1 x_1 + C_2 x_2$, so the general solution to this equation is

$$x(t) = x_h + x_p = C_1 e^t \cos t + C_2 e^t \sin t + \frac{1}{2}t + 1.$$

(c) Find the solution to the IVP given by the differential equation in (b) and initial conditions $x(0) = 0$, $x'(0) = 0$.
SOLUTION: First, evaluate at $t = 0$ and obtain

$$x(0) = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 + 0 + 1 = C_1 + 1 = 0.$$

Next differentiate $x(t)$ to obtain (since we know that $C_1 = -1$)
$x' = -e^t \cos x + e^t \sin x + C_2 \left(e^t \sin t + e^t \cos t\right) + \frac{1}{2}$, so that $y'(0) = C_2 \cdot 0 + C_2 e^0 \cos 0 + \frac{1}{2} = 0$.
It follows that $-1 + C_2 + 1/2 = 0$, so that $C_2 = 1/2$ and the solution to the IVP is

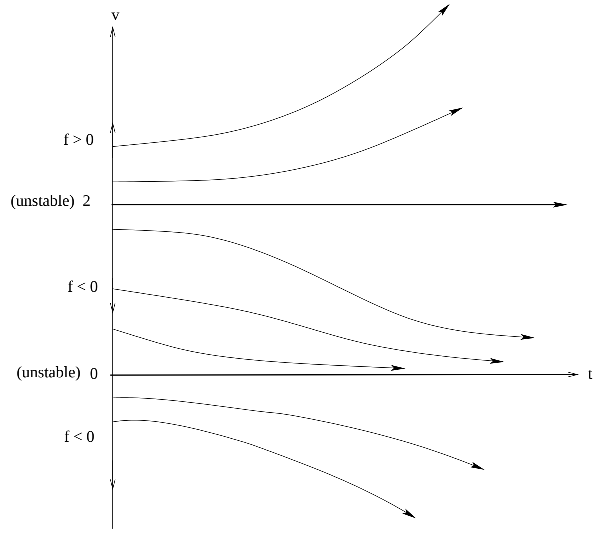
$$x = -e^t \cos t + \frac{1}{2} e^t \sin t + \frac{1}{2} t + 1.$$

(d) Convert the homogeneous system of (a) to a first order linear system, graph a typical solution in the phase plane, and classify the equilibrium point $(0, 0)$. (Hint: sketch a simple solution.)
SOLUTION: As a linear system we set $y = x'$ so that $y' = x'' = 2x' - 2x$, and obtain the system
$x' = y,\ y' = -2x + 2y$. From (a) we know that a simple solution is
$x(t) = e^t \cos t$ and $y(t) = e^t(\cos t - \sin t)$.
Thus both x and y oscillate between positive and negative and grow exponentially in size. This describes a spiral that sends solutions away from the origin, so $(0, 0)$ is an unstable spiral.
[Insert graph of counterclockwise unstable spiral.]

(16) **5**. Sketch the phase line for the differential equation $y' = y^2(y-2)$ along a vertical y-axis, classify the equilibrium solutions and then sketch a few representative solutions in the ty-plane.

SOLUTION:

The critical points are given by $0 = y^2(y-2) = f(y)$, so that $y = 0, 2$. Note that for $y < 0$, $f(y) < 0$, while for $0 < y < 2$, also $f(y) < 0$, and for $2 < y$, $f(y) > 0$. The corresponding equilibrium solutions are sketched below with the phase line, showing that both $y = 0$ and $y = 2$ are unstable.



(16) **6**. At time $t = 0$ a tank of capacity 400 gallons contains 200 gallons of brine solution which contains 100 ounces of salt. Four gallons of pure water is poured into the tank per minute, the brine is continuously mixed and leaves the tank at the constant rate of 2 gallons per minute. Express the problem of finding $S(t)$, the amount of salt in the tank at time t, as an IVP.

SOLUTION: The volume of water in the tank at time t is $V = 200 + (4-2)t$, where t is time. So the rate of change in total salt is given by

$\frac{dS}{dt} = \text{Rate in } - \text{Rate out } = 0 - \frac{S}{V}(\text{oz/gal}) \cdot 2\text{gal/min} = -\frac{2S}{200+2t} = -\frac{S}{100+t}$.

So the IVP is $\frac{dS}{dt} = -\frac{S}{t+100}$, $S(0) = 100$.

Just for the record, separation of variables leads to $\ln|S| = -\ln(t+100) + C$, so that $S = \frac{A}{t+100}$, and evaluation at $t = 0$ gives $A = 10000$, so $S = \frac{10{,}000}{t+100}$.

(14) **7**. Given that the linear system $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with constant coefficient matrix A has eigenvalue $-1+2i$, with a corresponding eigenvector $\begin{bmatrix} i \\ 2-i \end{bmatrix}$, find the general solution to the system.

SOLUTION: We know that a complex solution is given by

$$\mathbf{x}(t) = \begin{bmatrix} i \\ 2-i \end{bmatrix} e^{(-1+2i)t} = \begin{bmatrix} i \\ 2-i \end{bmatrix} e^{-t}(\cos 2t + i \sin 2t) = e^{-t} \begin{bmatrix} -\sin 2t + i \cos 2t \\ 2\cos 2t + \sin 2t + i(2\sin 2t - \cos 2t) \end{bmatrix}.$$

Taking real and imaginary parts gives two independent real solutions, so that the general solution is

$$\mathbf{x}(t) = C_1 \begin{bmatrix} -\sin 2t \\ 2\cos 2t + \sin 2t \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} \cos 2t \\ 2\sin 2t - \cos 2t \end{bmatrix} e^{-t} = e^{-t} \begin{bmatrix} -C_1 \sin 2t + C_2 \cos 2t \\ C_1(2\cos 2t + \sin 2t) + C_2(2\sin 2t - \cos 2t) \end{bmatrix}$$

(22) **8**. (a) Use the method of eigenvalues to find a general solution to the linear system $x_1' = x_1 + 2x_2$, $x_2' = 2x_1 + x_2$.
SOLUTION:
The system can be written as $\mathbf{x}' = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P\mathbf{x}$. The eigenvalues of the coefficient matrix are given by
$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$, so $\lambda = -1, 3$ are the eigenvalues.
For eigenvalue $\lambda = 3$, eigenvector $[a, b]^T$ satisfies

$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2a+2b \\ 2a-2b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so that $a = b$ and $\mathbf{v}_1 = [1, 1]^T$ is an eigenvector.
For eigenvalue $\lambda = -1$, eigenvector $[a, b]^T$ satisfies

$$\begin{bmatrix} 1--1 & 2 \\ 2 & 1--1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a+2b \\ 2a+2b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so that $a = -b$ and $\mathbf{v}_2 = [-1, 1]^T$ is an eigenvector.
Therefore, the general solution to the system is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1\mathbf{v}_1 e^{3t} + C_2\mathbf{v}_2 e^{-t} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} = \begin{bmatrix} C_1e^{3t} - C_2e^{-t} \\ C_1e^{3t} + C_2e^{-t} \end{bmatrix}.$$

(b) Use the solution in (a) to find the solution to the IVP consistsing of this system together with initial conditions $x_1(0) = -1$, $x_2(0) = 0$.
SOLUTION:
Evaluate the solution at $t = 0$ and use $e^{-3\cdot 0} = e^0 = 1 = e^{-0}$ to obtain

$$\mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} C_1 - C_2 \\ C_1 + C_2 \end{bmatrix}$$

so that $C_1 = -C_2$ and $C_1 - C_2 = -1 = -2C_2$, so that $C_2 = 1/2$ and $C_1 = -1/2$. Thus

$$\mathbf{x}(t) = -\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} = -\frac{1}{2}\begin{bmatrix} e^{3t} + e^{-t} \\ e^{3t} - e^{-t} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}e^{3t} - \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

(12) **9**. Compute the following Laplace transforms. Identify which table entries you use.
(a) $f(t) = (t+1)^2 + e^{2t+3}$, $\mathcal{L}\{f(t)\}(s) =$
SOLUTION: $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 + 2t + 1\} + e^3\mathcal{L}\{e^{2t} \cdot 1\} = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} + \frac{e^3}{s-2}$ by table items 3, 5, and 6.

(b) $f(t) = u(t-2)t^2 + \cos 4t$, $\mathcal{L}\{f(t)\}(s) =$
SOLUTION: $\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t-2)t^2\} + \mathcal{L}\{\cos 4t\} = e^{-2s}\mathcal{L}\left\{(t+2)^2\right\} + \frac{s}{s^2+16} = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right) + \frac{s}{s^2+16}$ by table items 12, 5, 6 and 8.

(16) **10**. Compute the following inverse Laplace transforms. Identify which table entries you use.

(a) $Y(s) = \dfrac{1}{s^2+2s-3}$, $\mathcal{L}^{-1}\{Y(s)\}(t) =$

SOLUTION:

$$\frac{1}{s^2+2s-3} = \frac{1}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}.$$

Clear denominators by multiplying by $(s+3)(s-1)$ and obtain
$1 = A(s-1) + B(s+3)$, so that at $s = 1$, get $B = 1/4$ and at $s = -3$, get $A = -1/4$. So

$$\mathcal{L}^{-1}\{Y(s)\}(t) = -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}(t) + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) = -\frac{1}{4}e^{-3t} + \frac{1}{4}e^{t}$$

by table entries 3 and 5.

(b) $Y(s) = \dfrac{2}{s(s+2)^2}$, $\mathcal{L}^{-1}\{Y(s)\}(t) =$

SOLUTION: We have

$$\frac{2}{s(s+2)^2} = \frac{1}{s}\frac{2}{s(s+2)^2} = \mathcal{L}\{1\}\mathcal{L}\{2e^{-2t}t\}$$

by table items 3,5 and 6, so by table item 15

$$\mathcal{L}^{-1}\left\{\frac{2}{s(s+2)^2}\right\} = \int_0^t 1\cdot 2e^{-2s}s\,ds = \left(-se^{-2s} - \frac{1}{2}e^{-2s}\right)\Big|_{s=0}^{t} = \frac{1}{2} - \left(t+\frac{1}{2}\right)e^{-2t}.$$

[Or in partial fractions, $\frac{2}{s(s+1)^2} = \frac{1}{2s} - \frac{1}{2(s+2)} - \frac{1}{(s+2)^2}$.]

(18) **11**. Use Laplace transforms to solve the IVP $y'' - 3y' + 2y = 1$, $y(0) = 0$, $y'(0) = 1$.

SOLUTION: Apply $\mathcal{L}$ to both sides and obtain

$$\begin{aligned}
\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= \mathcal{L}\{t\} \\
s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 3(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} &= \frac{1}{s} \\
\mathcal{L}\{y\} &= \frac{1+s}{s(s^2-3s+2)} = \frac{1+s}{s(s-1)(s-2)}
\end{aligned}$$

Partial fractions give
$\frac{1+s}{s(s^2-3s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$. Clear denominators and get that
$1 + s = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$, so that $s = 0$ gives $A = 1/2$, $s = 1$ gives $B = -2$, and $s = 2$ gives $C = 3/2$. Hence

$$y = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s} - \frac{2}{s-1} + \frac{3}{2}\frac{1}{s-2}\right\} = \frac{1}{2} - 2e^{t} + \frac{3}{2}e^{2t}.$$

(8) **12**. Compute the Laplacian of the function $f(t)$ given by

$$f(t) = \begin{cases} t, & 1 \le t < 4 \\ 0, & \text{otherwise} \end{cases}$$

SOLUTION: Form the impulse function $u(t-1) - u(t-3)$ which is one inside the interval $1 \le x < 4$ and zero outside it. Then $f(t) = (u(t-1) - u(t-4))\,t$. So by table item 12,

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\{u(t-1)\,t\} - \mathcal{L}\{u(t-4)\,t\} \\
&= e^{-s}\mathcal{L}\{t+1\} - e^{-4s}\mathcal{L}\{t+1\} \\
&= \left(e^{-s} - e^{-4s}\right)\left(\frac{1}{s^2} + \frac{1}{s}\right).
\end{aligned}$$