Sample Exam 2 **Math 221H** **Fall 2008**

Name:____________________________ **Score:**__________

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Make all obvious simplifications, e.g., 0 rather than $\sin\pi$. Point values of problems are given in parentheses. Point values of problems are given in parentheses. Notes or text in *any* form not allowed. No electronic equipment allowed.

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(14) **1.** Find general solutions to the following DEs.

(a) $y'' + 8y' + 25y = 0$

Solution. The characteristic equation for this homogeneous equation is

$$r^2 + 8r + 25 = 0,$$

with roots from the quadratic formula as

$$r = \frac{\left(-8 \pm \sqrt{64 - 4 \cdot 25}\right)}{2} = 4 \pm 3\text{i},$$

so that independent solutions are $e^{-4x}\cos 6x$ and $e^{-4x}\sin 6x$ and the general solution is

$$y_c(x) = c_1 e^{-4x}\cos 3x + c_2 e^{-4x}\sin 3x.$$

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(b) $(D-2)\left(D^2+9\right) y = 0$

Solution. Characteristic equation for this homogeneous equation is

$$(r-2)\left(r^2+9\right) = 0,$$

with roots $r = 2$ and $r = \pm\sqrt{-9} = \pm 3i$, so that independent solutions are e^{2x}, $\cos 3x$ and $\sin 3x$ and the general solution is

$$y_c(x) = c_1 e^{2x} + c_2\cos 3x + c_3 \sin 3x.$$

(17) **2.** Solve the IVP $3y^{(3)} + 2y'' = 0$, $y(0) =$-1 , $y'(0) = 0$, $y''(0) = 1$.

Solution. First solve the homogeneous equation. The characteristic equation for this problem is

$$0 = r^3 + 2r^2 = r^2(3r+2),$$

which has roots $r = -2/3, 0, 0$. It follows that a set of independent solutions for this problem is 1, x, e^{-2x}, the first two being for the repeated root 0. So the general solution and its first two derivatives are

$$\begin{aligned} y(x) &= c_1 + c_2 x + c_3 e^{-\frac{2}{3}x} \\ y'(x) &= c_2 - \frac{2}{3}c_3 e^{-\frac{2}{3}x} \\ y''(x) &= \frac{4}{9}c_3 e^{-2x}. \end{aligned}$$

Apply the initial conditions to obtain

$$\begin{aligned} -1 = y(0) &= c_1 + c_3 \\ 0 = y'(x) &= c_2 - \frac{2}{3}c_3 \\ 1 = y''(x) &= \frac{4}{9}c_3, \end{aligned}$$

so that $c_3 = 9/4$, then $c_2 = \frac{2}{3}c_3 = 3/2$, then $c_1 = -1 - c_3 = -13/4$, and the solution is

$$y(x) = -\frac{13}{4} + \frac{3}{2}x + \frac{9}{4}e^{-2x}.$$

(20) **3.** Given the solution $y_1(x) = e^{-2x}$ to the DE $y'' + 4y' + 4y = 0$, use reduction of order to find another solution and verify that the two solutions are linearly independent.
SOLUTION. We set $y_2(x) = v(x)\, y_1(x)$ and obtain derivatives, then multiply by appropriate coefficients to obtain

$$\begin{aligned} 4[y_2 &= ve^{-2x}] \\ 4[y_2' &= v'e^{-2x} - 2ve^{-2x}] \\ 1[y_2'' &= v''e^{-2x} - 4v'e^{-2x} + 4ve^{-2x}] \end{aligned}$$

Now add both sides and set the left hand side to zero, obtaining

$$0 = 4ve^{-2x} + 4v'e^{-2x} - 8ve^{-2x} + v''e^{-2x} - 4v'e^{-2x} + 4ve^{-2x} = v''e^{-2x}.$$

Hence $v'' = 0$, so $v' = 1$ and $v = x$ as specific antiderivatives. Thus, $y_2 = xe^{-2x}$.
To check for linear independence, we calculate the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix} = e^{-2x}\left(e^{-2x} - 2xe^{-2x}\right) - \left(-2e^{-3x}\right)xe^{-2x} = e^{-4x} \neq 0.$$

Since the Wronskian does not vanish (anywhere), these functions are linearly independent on the whole real line.
(18) **4.** Use the method of undertermined coefficients to find particular solutions to these DEs
(a) $y'' + 8y' + 25 = 0$ (In this part also write out the general solution to the equation.)
SOLUTION. This equation can be written in the form $y'' + 8y' = -25$. The characteristic equation for the associated homogeneous problem $y'' + 8y' = 0$ is $0 = r^2 + 8r = r(r+8)$, so the general solution for the homogeneous equation is $y_c(x) = c_1 1 + c_2 e^{-8x}$.
Also, the family of derivatives of $f(x) = -25$ is generated by the single function $g(x) = 1$, but this is a solution to the homogeneous equation. Therefore, multiply by x to obtain non-solutions to the homogeneous equation. Thus, we guess a particular solution of the form $y_p(x) = Ax$. Plug into the DE and obtain that $y_p'' + 8y_p' = 0 + 8A = -25$, and therefore $A = -25/8$. It follows that $y_p(x) = -\frac{25}{8}x$ and the general solution to the DE is

$$y(x) = y_c(x) + y_p(x) = c_1 + c_2 e^{-8x} - \frac{25}{8}x.$$

(b) $y''' - y'' - 12y' = x^2 e^{-3x}$ (In this part only set up the form, and do *not* solve for coefficients.)
SOLUTION. The characteristic equation for associated homogeneous problem $y''' - y'' - 12y' = 0$ is

$$0 = r^3 - r^2 - 12r = r\left(r^2 - r - 12\right) = r(r+3)(r-4)$$

with roots $r = 0, -3, 4$, so the general solution for the homogeneous equation is

$$y_c(x) = c_1 + c_2 e^{-3x} + c_3 e^{4x}.$$

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Also, the family of derivatives of $f(x) = x^2 e^{-3x}$ has $f' = 2xe^{-3x} - 3x^2 e^{-3x}$, and $f'' = 2e^{-3x} - 12xe^{-3x} + 9x^2 e^{-3x}$, etc., so that the functions

$$\left\{x^2 e^{-3x}, xe^{-3x}, e^{-3x}\right\}$$

span all derivatives. However, e^{-3x} is a solution to the homogeneous problem, so we must multiply this family by x to obtain non-solutions to the homogeneous problem. Therefore, our guess for a particular solution to the problem is

$$y_p(x) = Ax^3 e^{-3x} + Bx^2 e^{-3x} + Cxe^{-3x}.$$

(16) **5.** Use variation of parameters to find a particular solution to the DE $y'' + 9y = 2\sec 3x$.
SOLUTION. First find the solution to the associated homogeneous equation $y'' + 9y = 0$, which has characteristic equation $r^2 + 9 = 0$ and roots $r = \pm\sqrt{-9} = \pm 3i$. So independent solutions are

$$y_1(x) = \cos 3x, \quad y_2(x) = \sin 3x.$$

Next calculate

$$W(y_1, y_2) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\left(\cos^2 3x + \sin^2 3x\right) = 3.$$

Thus with right-hand side $f(x) = 2\sec 3x$,

$$\int \frac{y_1 f}{W} dx = \int \frac{\cos 3x \cdot \frac{2}{\cos 3x}}{3} dx = \frac{2}{3}$$

and

$$-\int \frac{y_2 f}{W} dx = -\int \frac{\sin 3x \cdot \frac{2}{\cos 3x}}{3} dx = -\frac{2}{3}\int \tan 3x \, dx = -\frac{2}{9}\ln|\sec 3x|,$$

so a particular solution is given by

$$\left\{-\int \frac{y_2 f}{W} dx\right\} y_1(x) + \left\{\int \frac{y_1 f}{W} dx\right\} y_2(x) = -\frac{2}{9}\ln|\sec 3x| \cdot \cos 3x + \frac{2}{3}\sin 3x.$$

(15) **6.** Consider the IVP $y'' + 8xy' + 25y = x^3$, $y(0) = 1$, $y'(0) = 2$.
(a) Convert this second order DE to a first order system.) Is this system linear? Homogeneous?
SOLUTION. Make the change of variables

$$\begin{aligned} y_1 &= y \\ y_2 &= y' \end{aligned}$$

so that

$$y_2' = y'' = -8xy' - 25y + x^3 = -8xy_2 - 25y_1 + x^3$$

and we have the first order sytem

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -25y_1 - 8xy_2 + x^3. \end{aligned}$$

This system is linear and non-homogeneous.
(b) Convert the initial conditions to a single initial condition for the system in (a)
SOLUTION. Since $y_2 = y'$, the initial condition for the system of (a) is

$$\begin{aligned} y_1(0) &= 1 \\ y_2(0) &= 2 \end{aligned}$$

(c) Give a reason why we might like to work with the IVP consisting of (a) and (b).
SOLUTION. One reason is we have good existence/uniqueness theorems for first order IVPs. Another is that we can use Euler's method to solve first order systems (with much the same formula as with a first order equation: $\mathbf{y}_{n+1} = \mathbf{y}_n + h \cdot \mathbf{f}(x_n, \mathbf{y}_n)$.)