Sample Exam 4 Key **Math 208** **Fall 2009**

Name:______________________________ **Score:**__________

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Make all obvious simplifications, e.g., 0 rather than $\sin \pi$. Point values of problems are given in parentheses. Point values of problems are given in parentheses. Notes or text in *any* form not allowed. The only electronic equipment allowed is a calculator.

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(28) **1.** Let S be the portion of the cone $z = 2\sqrt{x^2+y^2}$ between the planes $z = 2$ and $z = 4$ and vector field $\mathbf{F} = \langle 0, -x, z \rangle$.

(a) Determine whether or not the vector field $\mathbf{F}$ is conservative.

SOLUTION. We calculate

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -x & z \end{vmatrix} = \left\langle \frac{\partial z}{\partial y} - \frac{\partial(-x)}{\partial z}, -\left(\frac{\partial z}{\partial x} - \frac{\partial 0}{\partial z}\right), \frac{\partial(-x)}{\partial x} - \frac{\partial 0}{\partial z} \right\rangle = \langle 0, 0, -1 \rangle .$$

It follows that $\mathbf{F}$ is not conservative, since its curl does not vanish.

(b) Find a parametrization of S and express $\mathbf{r}$ (position vector) and $\mathbf{F}$ in terms of it.

SOLUTION. We note that the shadow of S on the xy-plane is the region R which is the annulus between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. We parametrize with polar coordinates r, θ and obtain

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \quad 1 \le r \le 2,\ 0 \le \theta \le 2\pi \\ z &= 2r \end{aligned}$$

so that $\mathbf{r} = \langle r\cos\theta, r\sin\theta, 2r \rangle$ and $\mathbf{F} = \langle 0, -r\cos\theta, 2r \rangle$. (Rectangular coordinates x, y would yield $\mathbf{r} = \left\langle x, y, 2\sqrt{x^2+y^2} \right\rangle$, where (x, y) is in the region R, and $\mathbf{F} = \left\langle 0, -x, 2\sqrt{x^2+y^2} \right\rangle$.)

(c) Set up (do not solve) an iterated integral for $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{n}$ is the upward pointing normal.

SOLUTION. We already have parametrized S, so next step is to find $\mathbf{n}\, d\sigma$:

$$\mathbf{n} = \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left\langle 0 - 2r\cos\theta, -(0 - 2(-r\sin\theta)), r\cos^2\theta - -r\sin^2\theta \right\rangle ,$$

so $\mathbf{n}\, d\sigma = \pm \langle -2r\cos\theta, -2r\sin\theta, r \rangle\, dA$. We choose $+$ for upward normal. Hence

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \int_0^{2\pi} \int_1^2 \langle 0, -r\cos\theta, 2r \rangle \cdot \langle -2r\cos\theta, -2r\sin\theta, r \rangle\, dr\, d\theta \\ &= \int_0^{2\pi} \int_1^2 (2\cos\theta\sin\theta + 2)\, r^2 dr\, d\theta. \end{aligned}$$

(16) **2.** Find a potential function for the vector field $\mathbf{F}(x,y) = \langle x-5, 3y^2+7\rangle$.

SOLUTION. We assume there is one, so that $\mathbf{F} = \langle f_x, f_y\rangle$, from which it follows that

$$f_x = x - 5, \quad f_y = 3y^2 + 7.$$

So integrate the first to obtain

$$f = \int f_x dx = \int (x-5)\,dx = \frac{x^2}{2} - 5x + C(y).$$

Now differentiate this expression to obtain

$$f_y = 0 + C'(y) = 3y^2 + 7.$$

Integrate again to obtain

$$C(y) = \int C(y)\,dy = \int \left(3y^2+7\right) dy = 3\frac{y^3}{3} + 7y + D = y^3 + 7y + D.$$

Hence

$$f(x,y) = \frac{x^2}{2} - 5x + y^3 + 7y + D$$

where D is an arbitrary constant (or simply $f(x,y) = \frac{x^2}{2} - 5x + y^3 + 7y$.

(20) **3.** Use Green's Theorem to evaluate $\oint_C \left(e^{x^2} - 2y\right) dx + \left(e^{y^2} + 4x\right) dy$, where C is the circle $x^2 + y^2 = 4$, oriented counterclockwise.

SOLUTION. The flux form of Green's Theorem is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA = \iint_R \nabla\times\mathbf{F}\cdot\mathbf{k}\,dA,$$

with C the positively oriented boundary of plane region R and

$$\mathbf{F} = \langle M, N\rangle = \left\langle e^{x^2} - 2y, e^{y^2} + 4x\right\rangle.$$

Calculate

$$N_x - M_y = \frac{\partial}{\partial x}\left(e^{y^2} + 4x\right) - \frac{\partial}{\partial y}\left(e^{x^2} - 2y\right) = 4 - (-2) = 6.$$

Thus

$$\oint_C \left(e^{x^2} - 2y\right) dx + \left(e^{y^2} + 4x\right) dy = \iint_R 6\,dA = 6\iint_R dA.$$

But the last double integral is just the area of a circle of radius 2. Hence

$$\oint_C \left(e^{x^2} - 2y\right) dx + \left(e^{y^2} + 4x\right) dy = 6\pi 2^2 = 24\pi.$$

(Or one could use the flux form of Green's Theorem: $\oint_C N\,dx - M\,dy = \iint_R (M_x + N_y)\,dA$.)

(18) **4.** Use Stokes' Theorem to express the flux integral $\iint_S \nabla \times (y\mathbf{i}) \cdot \mathbf{n}\, d\sigma$ as a definite integral (do not solve it), where S is the portion of the paraboloid $z = 1 - x^2 - y^2$above the xy-plane with outward pointing normal $\mathbf{n}$.

SOLUTION. Stokes' Theorem says that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma,$$

where closed curve C is the boundary of of S positively oriented with respect to the orientation $\mathbf{n}$ of S.
In this problem we take $\mathbf{F} = \langle y, 0, 0\rangle$ and the boundary of S is obtained by setting $z = 0$ and obtaining $0 = 1 - x^2 - y^2$. This is just the circle of radius 1, center at the origin, which is oriented counterclockwise to be positively oriented with respect to $\mathbf{n}$. This curve is parametrized as

$$\begin{aligned} x &= \cos t \\ y &= \sin t \qquad 0 \le t \le 2\pi \\ z &= 0, \end{aligned}$$

so that

$$\begin{aligned} dx &= -\sin t\, dt \\ dy &= \cos t\, dt \\ dz &= 0. \end{aligned}$$

Hence,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C y\, dx = \int_0^{2\pi} \sin t\, (-\sin t)\, dt = -\int_0^{2\pi} \sin^2 t\, dt.$$

(18) **5.** Use the Divergence Theorem to evaluate $\int\int_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle y^3 - 2x, e^{xz}, 4z\rangle$ and S is the boundary of the rectangular box $0 \le x \le 2$, $1 \le y \le 2$, $-1 \le z \le 2$, with exterior unit normal.

SOLUTION. The Divergence Theorem says that

$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV,$$

where $S = \partial D$ is the boundary surface of the solid D.
In this case

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\left(y^3 - 2x\right) + \frac{\partial}{\partial y} e^{xz} + \frac{\partial}{\partial z} 4z = -2 + 0 + 4 = 2.$$

Hence

$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D 2\, dV = 2 \iiint_D dV,$$

where $\iiint_D dV$ is just the volume of a box with sides of length 2, 1 and $2 - -1 = 3$. Hence

$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 2 \cdot 2 \cdot 1 \cdot 3 = 12.$$