Sample Exam 2 Key **Math 208** **Fall 2009**

Name:______________________________ **Score:**____________

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Make all obvious simplifications, e.g., 0 rather than $\sin\pi$. Point values of problems are given in parentheses. Notes or text in *any* form are not allowed. The only electronic equipment allowed is a calculator.

(24) **1.** Let $f(x,y) = 8x^2 + 4x^2y + y^2 - 7$.

(a) Find all derivatives up to the second order.

Calculate the partials:

$f_x = 16x + 8xy$	$f_y = 4x^2 + 2y$
$f_{xx} = 16 + 8y$	$f_{yy} = 2$
$f_{xy} = 8x$	$f_{yx} = 8x$

(b) Find all critical points of f.

Set $0 = f_x = 16x + 8xy$ and $0 = 4x^2 + 2y$.

From the second equation $y = -2x^2$.

Plug into first and obtain $0 = 16x - 16x^3 = 16x(1 - x^2)$, so $x = 0, -1, 1$.

For each x obtain corresponding $y = 0, -2, -2$.

Critical points are $(0,0)$, $(-1,-2)$, $(1,-2)$.

(c) Use the second derivative test to classify the critical points of f.

The discriminant is $D_f = f_{xx}f_{yy} - (f_{xy})^2 = (16 + 8y)\,2 - (8x)^2 = 32 + 16y - 64x^2$.

$D_f(0,0) = 32$, so we have a local max/min. Since $f_{yy}(0,0) = 2 > 0$, f has a local minimum at $(0,0)$.

$D_f(-1,-2) = 0 - 64 \cdot 1 = -64 < 0$, so f has a saddle point at $(-1,-2)$.

$D_f(1,-2) = 0 - 64 \cdot 1 = -64 < 0$, so f has a saddle point at $(1,-2)$.

(22) **2.** Let $f(x,y) = xy$.
(a) Find the extrema of f subject to the constraint $x^2 + 2y^2 = 1$ by the method of Lagrange multipliers.

The Lagrange equations are $g(x,y) \equiv x^2 + 2y^2 - 1 = 0$ and

$$\langle y, x \rangle = \langle f_x, f_y \rangle = \nabla f = \lambda \nabla g = \langle \lambda g_x, \lambda g_y \rangle = \langle \lambda 2x, \lambda 4y \rangle .$$

So the system is

$$\begin{aligned} y &= 2\lambda x \\ x &= 4\lambda y \\ x^2 + 2y^2 &= 1. \end{aligned}$$

If $x = 0$, first equation implies $y = 0$, which contradicts the third equation.
Likewise, if $y = 0$, second equation implies $x = 0$, which again contradicts the third.
So neither is zero. Solve for λ in the first and second and obtain that $\frac{y}{2x} = \lambda = \frac{x}{4y}$, so $4y^2 = 2x^2$ and $x^2 = 2y^2$. Plug this into the third equation and get $2y^2 + 2y^2 = 4y^2 = 1$, so $y = \pm\frac{1}{2}$. For each such y, $x^2 = 2\frac{1}{4} = \frac{1}{2}$, so $x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$.
This gives 4 critical points, $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2}\right)$. Now evaluate:
$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \frac{1}{2\sqrt{2}}$, a MAX value.
$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) = -\frac{1}{2\sqrt{2}}$, a MIN value.
$f\left(\frac{-1}{\sqrt{2}}, \frac{1}{2}\right) = -\frac{1}{2\sqrt{2}}$, a MIN value.
$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right) = \frac{1}{2\sqrt{2}}$, a MAX value.

(b) What additional point(s) should you check to find the absolute extrema of f over the region $x^2 + 2y^2 \leq 1$?

Use EVT. We checked boundary points of this closed bounded region in (a). We should also check critical points of $f(x,y)$ that are interior to the region. In this case, $0 = f_x = y$ and $0 = f_y = x$ gives a single critical point $(0,0)$ in the interior.

(18) **3.** Express the volume of the solid bounded above by the paraboloid $z = x^2 + y^2$ and below by the rectangle $R : 0 \leq x \leq 1$, $0 \leq y \leq 1$ as a double integral and evaluate this integral.

The volume is given by the double integral

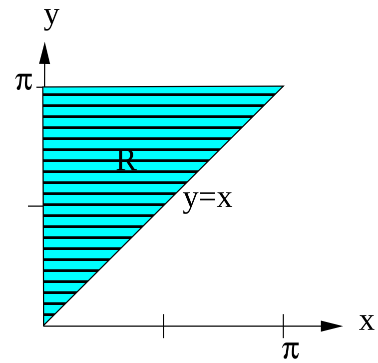
$$\begin{aligned} \iint_R \left(x^2 + y^2\right) dA &= \int_0^1 \int_0^1 \left(x^2 + y^2\right) dx\, dy \\ &= \int_0^1 \left(\frac{x^3}{3} + y^2 x\right)\Big|_{x=0}^1 dy \\ &= \int_0^1 \left(\frac{1}{3} + y^2\right) dy \\ &= \left(\frac{y}{3} + \frac{y^3}{3}\right)\Big|_{y=0}^1 = \left(\frac{1}{3} + \frac{1}{3}\right) = \frac{2}{3} \end{aligned}$$

(18) **4.** Evaluate the integral

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy\, dx$$

by interchanging the order of integration. Clearly sketch the region of integration.

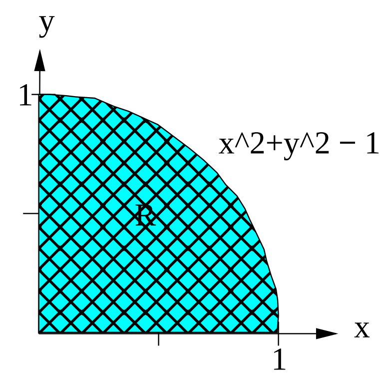
The region R is a triangle with vertices $(0,0)$, $(0,\pi)$ and (π,π). The integral is



$$\begin{aligned}
\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy\, dx &= \iint_R \frac{\sin y}{y} dA \\
&= \int_0^\pi \int_0^y \frac{\sin y}{y} dx\, dy \\
&= \int_0^\pi \frac{\sin y}{y} \int_0^y dx\, dy \\
&= \int_0^\pi \frac{\sin y}{y}\, x|_{x=0}^{y}\, dy \\
&= \int_0^\pi \frac{\sin y}{y} y\, dy \\
&= -\cos y|_0^\pi \\
&= \cos y|_\pi^0 = 1 - -1 = 2.
\end{aligned}$$

(18) **5.** Convert the iterated integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$ to polar coordinates and evaluate. Sketch the region of integration for this problem. What is the average value of $f(x,y) = x^2+y^2$ over this region?

The region R is the portion in the first quadrant of the disk with radius 1 and center at the origin. The integral is



$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx &= \iint_R (x^2+y^2)\, dA \\
&= \int_0^{\pi/2} \int_0^1 r^2 r\, dr\, d\theta \\
&= \int_0^{\pi/2} \left.\frac{r^4}{4}\right|_{r=0}^{1} d\theta \\
&= \int_0^{\pi/2} \frac{1}{4} d\theta \\
&= \frac{1}{4}\, \theta|_{\theta=0}^{\pi/2} \\
&= \frac{\pi}{8}.
\end{aligned}$$

The average value of f on R is $\bar{f}_R = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA$. In this case the area of a quarter circle of radius 1 is $\pi/4$ and we have calculated $\iint_R f(x,y)\, dA = \pi/8$, so $\bar{f}_R = \dfrac{1}{\pi/4}\dfrac{\pi}{8} = \dfrac{1}{2}$.