Name:____________________________ **Score:**________

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Point values of problems are given in parentheses. Notes or text in *any* form not allowed. The only electronic equipment allowed is a calculator.

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(15) **1.** Given points $P = (1, -1, 2)$, $Q = (2, 0, -1)$, $R = (0, 2, 1)$, $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$.

(a) Find $\mathbf{a} \times \mathbf{b}$ and $|\mathbf{a} \times \mathbf{b}|$. (Exercise 10.4.15)

Calculate $\mathbf{a} = \langle 2-1, 0-(-1), -1-2 \rangle = \langle 1, 1, -3 \rangle$ and $\mathbf{b} = \langle 0-1, 2-(-1), 1-2 \rangle = \langle -1, 3, -1 \rangle$, so

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = \langle 1(-1) - 3 \cdot (-3), -(1(-1) - 3(-1)), 1 \cdot 3 - (-1)1 \rangle = \langle 8, 4, 4 \rangle .$$

Also,

$$|\mathbf{a} \times \mathbf{b}| = |\langle 8, 4, 4 \rangle| = |4 \langle 2, 1, 1 \rangle| = 4\sqrt{2^2 + 1^2 + 1^2} = 4\sqrt{6}.$$

(b) Equation of the plane containing P, Q and R.

Since $\mathbf{a} \times \mathbf{b}$ is normal to the plane and P is on it, we may write the equation as

$$0 = 8(x-1) + 4(y+1) + 4(z-2) = 8x + 4y + 4z - 12,$$

or if we divide by 4, $2x + y + z = 3$.

(c) Parametric equations for a line through the point P and parallel to $\mathbf{a}$.

Since $\mathbf{a}$ is parallel to the line and P is on it, we may write the equations as

$$\begin{aligned} x &= 1 + 1t \\ y &= -1 + 1t, \ -\infty < t < \infty \\ z &= 2 - 3t. \end{aligned}$$

(15) **2.** Let $f(x, y) = y/x^2$. (Exercise 12.1.6)

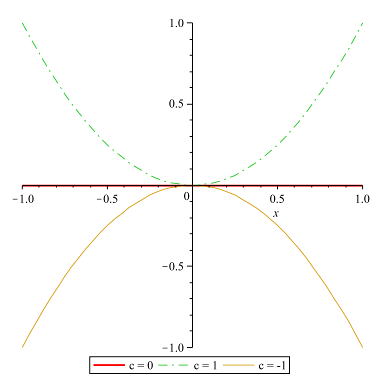
(a) Find the domain and range of f. Are these sets open or closed?

The function f is defined for all $x \neq 0$, so the domain of f is the set of all points (x, y) such that $x \neq 0$. This set is open but not closed.

Set $x = 1$ and let y vary arbitrarily and we see that the range of f is the set of all real numbers, which is both open and closed.

(b) Describe the contour curves of f and plot three of them.

The contours of f are the curves $y/x^2 = c$, where c is a constant. These are parabolas that exclude their common vertex, the origin, since $x = 0$ is not allowed. Thus, the graphs at the right for $c = -1, 0, 1$, have a "hole" at $(0, 0)$.



(c) At what points is $f(x, y)$ differentiable?

The partials of $f(x, y)$ are $\partial f/\partial x = -3y/x^2$ and $\partial f/\partial y = 1/x^2$, which are defined at all points in the domain of f, so f is differentiable at all points in its domain.

(17) **3.** Find the directional derivative of $f(x, y, z) = xy+yz+zx$ in the direction of $\mathbf{A} = \langle 3, 6, -2\rangle$at the point $P_0\,(1, -1, 2)$. In what direction from P_0 is the rate of greatest decrease of f greatest? (Exercise 12.5.13)

Here

$$f_x = y + z,\ f_y = x + z,\ f_z = y + x,$$

so that

$$\nabla f = \langle f_x, f_y, f_z\rangle = \langle y + z, x + z, y + x\rangle$$

and

$$\nabla f\,(P_0) = \langle -1 + 2, 1 + 2, -1 + 1\rangle = \langle 1, 3, 0\rangle\,.$$

A unit vector in the direction of $\mathbf{A}$ is

$$\mathbf{u} = \frac{1}{\sqrt{9 + 36 + 4}}\,\langle 3, 6, -2\rangle = \frac{1}{7}\,\langle 3, 6, -2\rangle\,.$$

Therefore, the directional derivative in the direction of $\mathbf{A}$ is

$$\left(\frac{df}{ds}\,(P_0)\right)_{\mathbf{u}} = \mathbf{u}\cdot\nabla f\,(P_0) = \frac{1}{7}\,\langle 3, 6, -2\rangle\cdot\langle 1, 3, 0\rangle = \frac{21}{7} = 3.$$

The direction of greatest decrease in f is the negative of the gradient, that is, the direction of the vector $\langle -1, -3, 0\rangle$.

(18) **4.** Let $f\,(x, y, z) = x^3z - 2yz^2 - 2z$. Find equations for the normal line and tangent plane to the surface $f(x, y, z) = 36$ at the point $(2, -1, 3)$. (Exercise 2, Gradient Applications)

Calculate

$$\nabla f = \left\langle 3x^2z, -2z^2, x^3 - 4yz - 2\right\rangle$$

and

$$\nabla f\,(2, -1, 3) = \left\langle 3\cdot 2^2\cdot 3, -2\cdot 3^2, 2^3 - 4\,(-1)\,3 - 2\right\rangle = \langle 36, -18, 18\rangle\,.$$

Since the normal line at $(2, -1, 3)$ is parallel to this vector, parametric equations are given by

$$\begin{aligned} x &= 2 + 36t \\ y &= -1 - 18t,\ -\infty < t < \infty \\ z &= 3 + 18t. \end{aligned}$$

Since the tangent plane has normal vector $\nabla f\,(2, -1, 3)$ and point $(2, -1, 3)$ on it, a defining equation is

$$0 = 36\,(x - 2) - 18\,(y + 1) + 18\,(z - 3) = 36x - 18y + 18z - 144,$$

or if we divide by 18, $2x - y + z = 8$.

(10) **5.** Given a function $w = h(x, y, z)$ with $x = f(u, v)$, $y = g(u, v)$ and $z = k(u, v)$, write a chain rule formula for $\partial fw/\partial u$ and $\partial w/\partial v$. (Exercise 12.4.15)

We have this classification:

Dependent variables:	w
Intermediate variables:	x, y, z
Independent variables:	u, v

So we use the simple chain rule with each intermediate and sum the results to obtain

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial h}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial u}\\ \frac{\partial w}{\partial v} &= \frac{\partial h}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial v}.\end{aligned}$$

(25) **6.** Let $w = f(x, y) = \sqrt{x^2 - y^2}$. (Exercise 5, total differentials handout)

(a) Compute the total differential of this function.

We have

$$\begin{aligned}dw &= f_x dx + f_y dy\\ &= \frac{2x\,dx}{2\sqrt{x^2 - y^2}} + \frac{(-2y)\,dy}{2\sqrt{x^2 - y^2}}\\ &= \frac{x\,dx - y\,dy}{\sqrt{x^2 - y^2}}.\end{aligned}$$

(b) Use the differential to estimate the largest possible error in computing $f(x, y)$ at $x = 5$ and $y = 3$, given that the error in x could be as large as 0.4 and the error in y could be as large as 0.2.
Use the differential approximation $\Delta w \approx dw$ at the point $(5, 3)$ and obtain

$$\Delta w \approx dw = \frac{5dx - 3dy}{\sqrt{25 - 9}} = \frac{5dx - 3dy}{4}.$$

By hypothesis, $|dx| \leq 0.4$ and $|dy| \leq 0.2$, so that the largest error is (approximately)

$$|\Delta w| \leq \frac{1}{4}|5dx - 3dy| \leq \frac{1}{4}(5|dx| + 3|dy|) = \frac{1}{4}(5 \cdot 0.4 + 3 \cdot 0.2) = \frac{2.6}{4} = 0.65.$$

(c) Compute the linearization $L(x, y)$ of f at $(5, 3)$ and use it to approximate $f(5, 2)$.
The linearization is given by

$$\begin{aligned}L(x, y) &= f(5, 3) + f_x(5, 3)(x - 5) + f_y(5, 3)(y - 3)\\ &= \sqrt{25 - 9} + \frac{5}{\sqrt{25 - 9}}(x - 5) - \frac{3}{\sqrt{25 - 9}}(y - 3)\\ &= 4 + \frac{5}{4}(x - 5) - \frac{3}{4}(y - 3)\\ &= \frac{1}{4}(16 + 5x - 25 - 3y + 9)\\ &= \frac{1}{4}(5x - 3y).\end{aligned}$$

Hence, we have

$$\sqrt{21} = f(5, 2) \approx L(5, 2) = \frac{1}{4}(25 - 6) = \frac{19}{4} = 4.75.$$