Exam 1 Key **Math 208** **Fall 2009**

Name:____________________________ **Score:**____________

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* Clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Make all obvious simplifications, e.g., 0 rather than $\sin \pi$. Point values of problems are given in parentheses. Notes or text in *any* form not allowed. The only electronic equipment allowed is a calculator.

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(18) **1.** Given points $P = (0, 2, -1)$ $Q = (-1, 0, 0)$, $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$. (Exer. 10.5.21)

(a) Find an equation for the plane through P and perpendicular (normal) to $\mathbf{n}$.

Solution. (5) This is given by

$$\begin{aligned} 0 &= \mathbf{n} \cdot \langle x - 0, y - 2, z - -1 \rangle \\ 0 &= \langle 3, -2, -1 \rangle \cdot \langle x - 0, y - 2, z - -1 \rangle \\ 0 &= 3x - 2(y - 2) - 1(z + 1) \\ 0 &= 3x - 2y - z + 3. \end{aligned}$$

(b) Find a vector orthogonal to both $\mathbf{a}$ and $\mathbf{n}$.

Solution. (4) This is given by the cross-product

$$\begin{aligned} \mathbf{v} &= \mathbf{a} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 1 \\ 3 & -2 & -1 \end{vmatrix} \\ &= ((-2)(-1) - 1(-2), -((-1)(-1) - 1 \cdot 3), ((-1)(-2) - (-2)3)) \\ &= \langle 4, 2, 8 \rangle \end{aligned}$$

(c) Find parametric equations for a line through the point P and parallel to $\mathbf{a}$.

Solution. (5) These are

$$\begin{aligned} x &= 0 - 1t = -t \\ y &= 2 - 2t \\ z &= -1 + t \end{aligned}$$

where t is any real number.

(d) Show that the line of (c) is contained in the plane of (a).

Solution. (4) Plug coordinates of a general point (or any two particular points) into the plane equation:

$$3(-t) - 2(2 - 2t) - z(-1 + t) + 3 = -3t + 4t - t - 4 + 1 + 3 = 0,$$

so the equation of the plane is satisfied.

(15) **2.** Let $f(x, y) = \sqrt{9 - x^2 - y^2}$. (Exer 12.1.8)

(a) Find the domain and range of f. Are these sets open, closed, or bounded?

Solution. (7) Need $9 - x^2 - y^2 \geq 0$, i.e., $x^2 + y^2 \leq 9$. So domain is

$$D = \left\{ (x, y) \mid x^2 + y^2 \leq 9 \right\},$$

which is a closed bounded set. The set of possible values goes goes from 0 to $\sqrt{9} = 3$, that is, the closed bounded interval $[0, 3]$.

(b) Describe the contour curves of f and plot three of them.

Solution. (5) From $z = \sqrt{9 - x^2 - y^2}$, get $x^2 + y^2 = 9 - z^2$, so these curves are circles of radius $\sqrt{9 - z^2}$. One could draw circles of radius 3, $\sqrt{5}$ and 0 (just a point) in the xy-plane and label them $z = 0, 2, 3$, resp.

2. (continued) (c) What is $\lim_{(x,y)\to(0,0)} f(x,y)$?
SOLUTION. (3) Since f is certainly continuous there (evaluation works on this algebraic expression), so the answer is

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = \sqrt{9-0} = 3.$$

(17) **3.** Find the directional derivative of $f(x,y) = -(x^2+y^2)/4$ at $P_0 = (-3,4)$ in the direction from P_0 toward $(0,1)$. In what direction from P_0 is the rate of increase of f the greatest? The least? Equal to zero? (Exer. 1, Gradient applications handout)
SOLUTION. We calculate

$$\nabla f(x,y) = \frac{-1}{4}\langle 2x, 2y\rangle = \frac{-1}{2}\langle x, y\rangle,$$

so that

$$\nabla f(P_0) = \frac{-1}{2}\langle -3.4\rangle = \frac{1}{2}\langle 3, -4\rangle.$$

A vector in the specified direction is

$$\langle 0 - -3, 1-4\rangle = \langle 3, -3\rangle = 3\langle 1, -1\rangle,$$

so a unit vector in that direction is $\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1,-1\rangle$and therefore the directional derivative in that direction is

$$\mathbf{u}\cdot\nabla f(-3,4) = \frac{1}{\sqrt{2}}\langle 1,-1\rangle\cdot\frac{1}{2}\langle 3,-4\rangle = \frac{1}{2\sqrt{2}}(3+4) = \frac{7}{2\sqrt{2}} = \frac{7}{4}\sqrt{2}\approx 2.47.$$

The direction of greatest increase of f is just $\nabla f(P_0) = \frac{1}{2}\langle 3,-4\rangle$.
The direction of least rate of increase of f is $-\nabla f(P_0) = \frac{1}{2}\langle -3,4\rangle$.
The direction of zero rate of increase of f is any vector orthogonal to $\nabla f(P_0)$, so $\frac{1}{2}\langle 4,3\rangle$ or its negative work for this.

(17) **4.** Let $f(x,y,z) = y^2 - x^2 - z$. What quadric is the level surface $f(x,y,z) = 0$? Find equations for the normal line and tangent plane to this surface at the point $(1,2,3)$.
SOLUTION. First, the surface $z = y^2 - x^2$ is a hyperbolic paraboloid.
Next, to find a normal, calculate $\nabla f = \langle -2x, 2y, -1\rangle$, so that a normal vector is

$$\nabla f(1,2,3) = \langle -2\cdot 1, 2\cdot 2, -1\rangle = \langle -2, 4, -1\rangle$$

and the equation of the tangent plane is

$$0 = \nabla f(1,2,3)\cdot\langle x-1, y-2, z-3\rangle = -2(x-1) + 4(y-2) - 1(z-3),$$

that is,

$$-2x + 4y - z - 3 = 0.$$

The normal line is given parametrically by

$$\begin{aligned} x &= 1-2t \\ y &= 2+4t \\ z &= 3-t \end{aligned}$$

(15) **5.** Given a function $z(s,t) = f(x(s,t), y(s,t))$, write a chain rule formula for z_t. Use this formula and the additional information that $x = 3st$, $y = 8s/t^2$, $f_x(-6,2) = 5$ and $f_y(-6,2) = -3$ to find $z_t(1,-2)$. (Exer. 4, Chain rule handout)

SOLUTION. The dependent variable is z, intermediate variables are x, y and independent variables are s, t. Thus the chain rule formula for $z_t = \partial z/\partial t$ is

$$z_t = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Now at $s = 1$ and $t = -2$ we have $x = 3 \cdot 1 \cdot (-2) = -6$, and $y = 8/(-2)^2 = 2$, so that

$$(x(1,-2), y(1,-2)) = (-6, 2).$$

Furthermore,

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}(3st) = 3s \quad \text{and} \quad \frac{\partial y}{\partial t} = \frac{\partial}{\partial t}\left(\frac{8s}{t^2}\right) = -2\frac{8s}{t^3} = -\frac{16s}{t^3}.$$

Therefore

$$z_t(1,-2) = f_x(-6,2)\, 3 \cdot 1 + f_y(-6,2)\frac{-16 \cdot 1}{(-2)^3} = 5 \cdot 3 - 3\frac{-16}{-8} = 15 - 6 = 9.$$

(18) **6.** Let $f(x,y) = \sqrt{y^2 - x^2}$. (Exer. 5, Total differentials handout)

(a) Compute the total differential of this function.

SOLUTION. (6) We have the total differential

$$dz = f_x dx + f_y dy = \frac{1}{2}\frac{-2x}{\sqrt{y^2 - x^2}}dx + \frac{1}{2}\frac{2y}{\sqrt{y^2 - x^2}}dy = \frac{-x\,dx + y\,dy}{\sqrt{y^2 - x^2}}.$$

(b) Given that your error in measuring x could be as large as 0.4 and the error in measuring y could be as large as 0.2, use differentials to estimate the possible error in evaluating $f(x,y)$ as $f(12,13)$.

SOLUTION. (6) Evaluate the differential at $x = 12$, $y = 13$, let dx and dy be the error in x and y, and obtain that the error in $z = f(x,y)$, $\triangle z$, satisfies

$$\triangle z \approx dz = \frac{-12\,dx + 13\,dy}{\sqrt{13^2 - 12^2}}.$$

Thus

$$|\triangle z| \approx \left|\frac{-12\,dx + 13\,dy}{5}\right| \leq \frac{1}{5}\{12|dx| + 13|dy|\} \leq \frac{1}{5}\{12 \cdot 0.4 + 13 \cdot 0.2\} = \frac{37}{25} = 1.48.$$

(c) Compute the linearization $L(x,y)$ of f at $(12,13)$.

SOLUTION. (6) Evaluate the differential at $x = 12$, $y = 13$, $dx = x - 12$, $dy = y - 13$, $dz = z - f(12,13) = z - 5$ and obtain

$$z - 5 = -\frac{12}{5}(x - 12) + \frac{13}{5}(y - 13),$$

so that

$$\begin{aligned} z &= L(x,y) = 5 + -\frac{12}{5}(x - 12) + \frac{13}{5}(y - 13) \\ &= 5 - \frac{12}{5}x + \frac{13}{5}y + \frac{1}{5}(169 - 144) \\ &= -\frac{12}{5}x + \frac{13}{5}y. \end{aligned}$$