Math 208
Triple integration exercises

1. Convert the integral $\int_0^2 \int_0^{2z} \int_y^{2y} f(x,y,z)\,dx\,dy\,dz$ to one in the order:

   a) $dz\,dx\,dy$

   b) $dx\,dz\,dy$

2. a) Prove that if a, b, c, d, e and k are constants and f, g and h are continuous functions of one variable, then

   $$\int_a^b \int_c^d \int_e^k f(x)g(y)h(z)\,dz\,dy\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)\left(\int_e^k h(z)\,dz\right)$$

   b) Apply the equality in part (a) to compute the value of $\int_0^3 \int_1^4 \int_{-1}^2 \frac{x(z^2-4)}{\sqrt{y}}\,dz\,dy\,dx$

3. Evaluate $\int_0^2 \int_0^{2y} \int_{z-2y}^0 3(x+2y-z)^{\frac{1}{2}}\,dx\,dz\,dy$

4. In each of the following, set up and evaluate a triple integral which gives the volume of the (unique!) finite region bounded by the surfaces whose equations are given.

   a) $z = y^2$, $z = x$ and $x = 4$

   b) $z = 0$, $z = x + 2y$, $z = 4 - x - 3y$ and $z = -y$

   c) $y = z^2$, $y = x^2$ and $z = 4$

5. (Challenging) Set up an iterated (triple) integral for the regions in problem 4 in as many different orders as possible.

Answers: 1. a) $\int_0^4 \int_y^{2y} \int_{y/2}^{2} f(x,y,z)\,dz\,dx\,dy$ b) $\int_0^4 \int_{y/2}^{2} \int_y^{2y} f(x,y,z)\,dx\,dz\,dy$

2. b) -81

3. $\frac{512}{35}$

4. a) $\int_{-2}^{2} \int_{y^2}^{4} \int_z^4 dx\,dz\,dy = \frac{256}{15}$ b) $\int_0^4 \int_{-z}^{4-2z} \int_{z-2y}^{4-z-3y} dx\,dy\,dz = \frac{32}{3}$

c) $\int_0^4 \int_{-z}^{z} \int_{x^2}^{z^2} dy\,dx\,dz = \frac{256}{3}$

5. a) Besides the iterated integral given in the solutions to 4 above, we have:

$$\int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_z^4 dx\,dy\,dz \text{ or } \int_0^4 \int_z^4 \int_{-\sqrt{z}}^{\sqrt{z}} dy\,dx\,dz \text{ or } \int_0^4 \int_0^x \int_{-\sqrt{z}}^{\sqrt{z}} dy\,dz\,dx \text{ or}$$

$$\int_{-2}^{2} \int_{y^2}^{4} \int_{y^2}^{x} dz\,dx\,dy \text{ or } \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} \int_{y^2}^{x} dz\,dy\,dx\,.$$

b) The integral in 4b is the only option for doing it as one iterated integral.

c) Besides the iterated integral given in 4c above, we have:

$$\int_{-4}^{4} \int_{x^2}^{16} \int_{\sqrt{y}}^{4} dz\,dy\,dx \text{ or } \int_0^{16} \int_{-\sqrt{y}}^{\sqrt{y}} \int_{\sqrt{y}}^{4} dz\,dx\,dy \text{ or } \int_0^4 \int_0^{z^2} \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz \text{ or}$$

$\int_0^{16} \int_{\sqrt{y}}^{4} \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy$; only the order $dy\,dz\,dx$ cannot be done