Math 208
Surface Classification

1. Refresher on curve types

First, you must know (this should have been in your precalculus course) how to tell whether a quadratic equation in two variables is a parabola, ellipse, hyperbola, or one of their degenerate forms. (A degenerate parabola is either one line or two parallel lines or the empty set; a degenerate ellipse is a point or the empty set; a degenerate hyperbola is a pair of lines that intersect.) The simplest forms of the equations are given in the following table, but with equations that are more complex, things like completing the square and substitution of variables may be necessary to make the equation recognizable. Note in all cases you can switch x and y without changing the type of curve given by the equation (e.g. $y = x^2$ and $x = y^2$ both give parabolas).

Common curve equations	
parabola	$y = ax^2 + bx + c, \quad a \neq 0$
ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$`
hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
hyperbola	$xy = c\ ,\ c \neq 0$
degenerate parabola ((parallel) straight line(s) or empty)	$ax^2 + bx + c = 0, \quad a \neq 0$
degenerate ellipse (one point)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$
degenerate ellipse (empty set)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = c, \quad c < 0$
degenerate hyperbola (intersecting straight lines)	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$
degenerate hyperbola (intersecting straight lines)	$xy = 0$

2. Classifying quadric surfaces using traces.

A trace of a surface is what you get by intersecting the surface with a plane. Usually, when dealing with traces, we look at planes of the form $x = c$ or $y = c$ or $z = c$, and only move on to other planes if these don't tell us enough. Most of the time with simple surfaces, in fact, we get enough information just by looking at the intersection of the surface with $x = 0$, $y = 0$, and $z = 0$.

Below, we describe the traces usually obtained from our quadric surfaces (surfaces coming from quadratic equations in 3 variables), if they're oriented optimally with respect to the axes. Note there is one type of quadratic equation that produces none of the surfaces listed below – that's what happens when a quadratic equation can be written as $(ax + by + cz + d)(ex + fy + gz + h) = 0$, where a, b, c, d, e, f, g and h are all constants. This factors as two linear equations, meaning it produces one or two planes (if two planes, they can be parallel or intersect). <u>All</u> traces in this case will consist of one or two straight lines or the entire plane.

Surface type	Surface subtype	Traces
ellipsoid		all traces are ellipses or degenerate ellipses
paraboloid	hyperbolic paraboloid	parabolas in 2 directions, hyperbolas or intersecting lines in the third direction.
paraboloid	parabolic cylinder	parabolas or degenerate parabolas or straight lines in all 3 directions
paraboloid	elliptic paraboloid	parabolas in 2 directions, ellipses (or circles) in the third direction
hyperboloid	hyperboloid of two sheets	hyperbolas (a few degenerate) in two directions, ellipses in the third direction, and the ellipses in some cases disappear
hyperboloid	elliptic cone	hyperbolas (a few degenerate) in two directions, ellipses in the third direction, and the ellipse for one trace shrinks to a point but never disappear
hyperboloid	hyperboloid of one sheet	hyperbolas (a few degenerate) in two directions, ellipses in the third direction, and the ellipses are never degenerate (always positive radii)

3. Classifying paraboloids using a formula

It turns out that paraboloids are easy to classify. Their equations can generally be rewritten as one variable equals a quadratic polynomial in the other two variables. (That means one variable starts out only in a linear term.) Suppose z starts out linear, then rewrite the equation so z equals a quadratic in x and y. Assume the coefficients are now given by

$$z = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

where A, B, C, D, E, and F are constants, and at least one of A, B, and C is non-zero (otherwise, we have a plane, not a quadratic).

Then the type of paraboloid is determined by the quantity $B^2 - 4AC$, which should be familiar from the quadratic formula. If $B^2 - 4AC > 0$, we have a hyperbolic paraboloid. If $B^2 - 4AC < 0$, we have an elliptic paraboloid. If $B^2 - 4AC = 0$, we have a parabolic cylinder.

Example: Given the equation $2x + 3x^2 - 10xz - y + 8z^2 = 14z + 9$, we see y is the variable that only appears in a linear term, so we solve for it in terms of x and z and rewrite the equation as $y = 3x^2 - 10xz + 8z^2 + 2x - 14z - 9$. Now $A = 3$, $B = -10$, C = 8, D = 2, E = −14, and F = −9. Thus $B^2 - 4AC = 100 - 96 > 0$, so this is a hyperbolic paraboloid.

P.S. If you're curious, the reason $B^2 - 4AC$ arises here is essentially the same reason it arises in the quadratic formula – it shows up in the quadratic formula when you complete the square to eliminate the linear term, here it shows up if you complete the square to eliminate the cross-product (*xy*) term.